Math 560
Fall 2005
Homework 7 Partial Solutions
Assigned Monday, 10 October, 2005

1. If f is uniformly continuous on a set S then f is continuous on S.

   Proof. Assume that f is uniformly continuous on S. Then for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x, y \in S$ and $\mid x - y \mid < \delta$, $\mid f(x) - f(y) \mid < \varepsilon$.

   To show that f is continuous, let $p_0 \in S$ and let $\varepsilon > 0$ be given. From uniform continuity, there is a $\delta > 0$ so that whenever $x, p_0 \in S$ with $\mid x - p_0 \mid < \delta$, $\mid f(x) - f(p_0) \mid < \varepsilon$. This is the definition of a function continuous at p_0. This works for all $p_0 \in S$, and so f is continuous on S. □

2. (Exercise #5a, p. 88) Let A and B be disjoint closed sets and suppose f is uniformly continuous on each. Show that f is uniformly continuous on $A \cup B$ if A is compact.

3. (Exercise #7, p. 88) Let D be a bounded set and let f be uniformly continuous on $D \subset \mathbb{R}^n$. Prove that f is bounded on D.

   Proof. Since f is uniformly continuous, for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $p, q \in D$ with $\mid p - q \mid < \delta$ then $\mid f(p) - f(q) \mid < \varepsilon$. In particular, this works for $\varepsilon = 1$.

   For a proof by contradiction, assume that f is not bounded, i.e. for every $M > 0$ there exists a $p \in D$ so that $f(p) > M$. Then let $p_1 \in D$ be so that $f(p_1) > 1$, $p_2 \in D$ with $f(p_2) > 2, \ldots, p_n \in D$ with $f(p_n) > n, \ldots$. Then $\{p_n\} \subset D$ is a bounded infinite set, and therefore has a limit point. Since it has a limit point, there is a subsequence, p_{n_k} which converges to that limit point. This new sequence is a convergent sequence, and therefore is Cauchy. Therefore, there is a N so that $\mid p_{n_k} - p_{n_j} \mid < \delta$ whenever $k, j > N$.

   Now the image of a Cauchy sequence under a uniformly continuous function is Cauchy. Therefore $\{f(p_{n_k})\}$ is a Cauchy sequence. Every Cauchy sequence is bounded. Therefore $\{f(p_{n_k})\}$ is bounded, but this sequence is unbounded. This is a contradiction. Therefore there is an M so that $f(p) \leq M \ \forall p$.

   □

4. (You should be able to do this, but I won't collect it) (Exercise #3, p. 88) Let f and g each be uniformly continuous on E. Show that $f + g$ is uniformly continuous on E.

   Proof. Assume that f and g are uniformly continuous. Let $\varepsilon > 0$ be given. Then $\frac{\varepsilon}{2} > 0$. Since f is uniformly continuous, there is a $\delta_f > 0$ so that if $x, y \in E$ and $\mid x - y \mid < \delta_f$ then $\mid f(x) - f(y) \mid < \frac{\varepsilon}{2}$. Similarly since g is uniformly continuous, there is a $\delta_g > 0$ so that whenever $x, y \in E$ and $\mid x - y \mid < \delta_g$, $\mid g(x) - g(y) \mid < \frac{\varepsilon}{2}$. Now let $\delta = \min\{\delta_f, \delta_g\}$. Then if $x, y \in E$ and $\mid x - y \mid < \delta$ we have:

   $$\mid f(x) + g(x) - f(y) - g(y) \mid \leq \mid f(x) - f(y) \mid + \mid g(x) - g(y) \mid < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

   Therefore $f + g$ is uniformly continuous. □