Math 560

Fall 2005

Homework 6 Partial Solutions

Assigned Monday, 3 October, 2005

1. (Be able to do this problem, but I will not collect it.) Prove Theorem 4 on page 77. That is, let f and g be continuous real-valued functions on a domain D. Show that the following functions are also continuous:

   (a) $f + g$

   (b) fg

   (c) $\alpha f + \beta g$ for $\alpha, \beta \in \mathbb{R}$

   (d) $\dfrac{f}{g}$ whenever $g(p) \neq 0$.

2. (#4, page 80) Let $f(x, y) = \dfrac{x^2y^2}{x^2 + y^2}$ with $f(0, 0) = 0$. By checking various sequences, test this for continuity at $(0, 0)$. Can you tell whether or not it is continuous there?

3. (#7, page 81) Use the example $f(x, y) = x^2$ to show that a continuous function does not have to map an open set onto an open set.

4. (#8, page 81) Use the example $f(x) = \dfrac{x^2}{1 + x^2}$ to show that a continuous function does not always have to map a closed set to a closed set.

5. (#10, page 81) Show that f is continuous iff the inverse images of closed sets are closed sets relative to D.

   Proof. Let $f : X \to Y$ be continuous. Let $K \subset Y$ be closed. Then $Y - K$ is open, and so $f^{-1}(Y - K) = f^{-1}(Y) - f^{-1}(K) = X - f^{-1}(K) = (f^{-1}(K))^c$. Since f is continuous, $(f^{-1}(K))^c$ is open. Therefore, $f^{-1}(K)$ is closed, as desired.

   To prove the converse, assume that the inverse image of closed sets is closed. Let $O \subset Y$ be open. Then $Y - O$ is closed. Therefore, $f^{-1}(Y - O) = f^{-1}(Y) - f^{-1}(O) = X - f^{-1}(O) = (f^{-1}(O))^c$ is closed (since the inverse image of closed sets is closed). Therefore $f^{-1}(O)$ is open. Therefore the inverse image of open sets is open, and f is continuous. $\square$

6. (#12, page 81) How are $f^{-1}(A \cap B)$ and $f^{-1}(A \cup B)$ related to $f^{-1}(A)$ and $f^{-1}(B)$? (Of course you need to prove your claims).

   See images handout.

7. If f is a continuous function from a compact subset $K \subset \mathbb{R}^n$ into $\mathbb{R}^m$, then $f(K)$ is compact in $\mathbb{R}^m$.

   Sieradski, p. 202. Let $\mathcal{O}$ be a covering of $f(K) \subset \mathbb{R}^m$ by open subsets of $\mathbb{R}^m$. By the continuity of f, $f^{-1}(\mathcal{O}) = \{f^{-1}(O) \subset \mathbb{R}^n : O \in \mathcal{O}\}$ is an open covering of $K \subset \mathbb{R}^n$. Since K is compact, it has a finite subcovering $\{f^{-1}(O_i) : 1 \leq i \leq k\}$. So $K \subseteq \cup_i f^{-1}(O_i) = f^{-1}(\cup_i O_i)$ and $f(K) \subseteq \cup_i U_i$. Thus the covering $\mathcal{O}$ of $f(A)$ has the finite subcovering $\{O_i : 1 \leq i \leq k\}$. Therefore $f(K)$ is compact. $\square$