Math 560
Fall 2005
Homework 5 Solutions
Assigned Wednesday, 22 September, 2005

1. Without using the Heine-Borel Theorem, Prove that the interval $[a, b]$ is compact.

   This is in your book.

2. Without using the Heine-Borel Theorem, Prove that a closed ball in $\mathbb{R}^k$ is compact.

3. Prove: If E s an infinite subset of a compact set K then E has a limit point in K.

   Proof from Rudin. In no point of K were a limit point of E then each $q \in K$ would have a neighborhood V_q which contains at most one point of E (namely q if $q \in E$). Then there is no finite subcollection of $\{V_q\}$ which can cover E; and the same is true of K since $E \subset K$. This contradicts the compactness of K. □

4. (Section 1.8, #5, p. 69) Prove that S in n space is compact if and only if every sequence in S has a limit point that belongs to S.

   Proof from Ross. Suppose that S is compact. Then it is closed and bounded. Take a sequence $\{p_n\} \subset S$. Since $\{p_n\}$ is bounded, there is a convergent subsequence (did we prove this? can you prove this?). So a subsequence of $\{p_n\}$ converges to some $p_0 \in \mathbb{R}^n$. Since S is closed, $p_0 \in S$.

   Suppose every sequence in S has a subsequence that converges to a point in S. Then it would suffice to show that S is closed and bounded (by the Heine-Borel Theorem). If S were unbounded, then S would contain a sequence $\{p_n\}$ such that $| p_n - 0 |= \infty$, and then no subsequence would converge. Thus S is bounded. If S were not closed then there would be a sequence of $\{p_n\} \subset S$ that would converge to $p_0 \notin S$. Since then every subsequence would converge to $p_0 \notin S$, we have a contradiction. Therefore, S is closed and bounded, hence compact. □