Math 560
Fall 2005
Homework 3 Partial Solutions
Assigned Wednesday, 7 September, 2005

1. (p. 28, #10) If f is a function of two variables, show that if the set of points above the graph of f is a convex set, then f is midpoint convex, i.e. it satisfies the equation:

$$f\left(\frac{1}{2}p + \frac{1}{2}q\right) \leq \frac{1}{2}f(p) + \frac{1}{2}f(q)$$

2. (p. 28, # 14) Given $x_n = 3n + (-1)^n(n-5) + 7$,

   (a) Find $x_1, x_2, \ldots, x_{10}$

   (b) Find all the numbers that ever appear twice in the entire sequence.

   (c) Do any terms appear three times?

   Solution

   (a)

$$x_1 = 14 \qquad x_6 = 26$$
$$x_2 = 10 \qquad x_7 = 26$$
$$x_3 = 18 \qquad x_8 = 34$$
$$x_4 = 18 \qquad x_9 = 30$$
$$x_5 = 22 \qquad x_{10} = 42$$

   (b) There are two important subsequences to consider - the even terms and the odd terms. The even terms are of the following form:

$$\begin{aligned} x_{2n} &= 3 \cdot 2n + (-1)^{2n}(2n-5) + 7 \\ &= 6n + 2n - 5 + 7 \\ &= 8n + 2 \end{aligned}$$

   The odd terms are of the form

$$\begin{aligned} x_{2n-1} &= 3(2n-1) + (-1)^{2n-1}(2n-1-5) + 7 \\ &= 6n - 2 - (2n-6) + 7 \\ &= 6n - 3 - 2n + 6 + 7 \\ &= 4n + 10 \end{aligned}$$

   By these explicit formulas, we can see that there are never two terms in the even subsequence which are the same, nor are there two terms in the odd subsequence which are the same. Therefore, the only way to get the same numbers out of the sequence more than once is for an even term to match an odd term, i.e. for

$$8n + 2 = 4m + 10$$
$$8n - 4m = 8$$
$$2n - m = 2$$

So that $2n = m + 2$
Therefore, the terms which yield the same numbers are those with $2n = m + 2$.

(c) There can never be terms that appear three times, since then at least two of the terms would have to have an even index or two of the terms would have to have an odd index.

3. (p. 28, #15) What is $\bigcup_1^\infty D_n$ where $D_n = \{p \in \mathbb{R}^n \mid \mid p \mid \leq n\}$?

4. (p. 28, #16) Show that the collection of all functions defined on a set D, with values in $\mathbb{R}^3$ is a vector space.

5. Prove that: The intersection of any number of closed sets is closed, but the union of an infinite number of closed sets need not be closed.

Proof. Let $\{F_\alpha\}$ be a family of closed sets. Then we want to look at $\bigcap F_\alpha$. To show that $\bigcap F_\alpha$ is closed, examine $(\bigcap F_\alpha)^c$ and show that the complement is open. Then since F_α is closed $\forall\alpha$, F_α^c is open $\forall\alpha$. Therefore,

$$\left(\bigcap F_\alpha\right)^c = \bigcup (F_\alpha^c)$$

is a union of open sets, which is open. Therefore $\bigcap F_\alpha$ is closed, as desired. □

Counterexample. Let $F_n = \left[\frac{1}{n}, 1\right]$. Then F_n is closed $\forall n$. However, $\bigcup F_n = (0, 1]$ which is not closed.

6. The interior of a set S is the largest open set that is contained in S.

7. (p. 37, #5) Let $S = \{(x, y) \mid x, y \in \mathbb{Q}\}$
   (a) What is the interior of S?
   (b) What is the boundary of S?

8. (p. 37, #7) Produce an unbounded set with no cluster points.

9. (p. 37 #10) Construct pictures to show that each of the following is false:
   (a) If $A \subset B$ then $bdy(A) \subset bdy(B)$
   (b) $bdy(S) = bdy(\overline{S})$
   (c) $bdy(S) = bdy(int(S))$
   (d) The interior of S is the same as the interior of the closure of S.

10. (p. 37, #15) Let $A, B \subset R^2$ be connected sets. Are $A \cap B$ and $A \cup B$ connected?