

Math 560
Fall 2005
Homework 1 - Partial Solutions
Assigned Wednesday, 24 August, 2005

1. Complete exercise 19 on page 11: Derive the following from the axioms of the real field:

- (a) For any $a, b \in \mathbb{R}$, if $a < b$ then $-b < -a$
- (b) For any $x \in \mathbb{R}$ $x^2 \geq 0$ with equality only if $x = 0$
- (c) For any $x, y \in \mathbb{R}$, if $x^2 + y^2 = 0$ then $x = y = 0$.

2. Complete exercise 20 on page 11:

- (a) If $0 < m < n$ then $m^2 < n^2$ and $\frac{1}{n} < \frac{1}{m}$
- (b) If $m < n < 0$ show that $m^2 > n^2$ and $\frac{1}{m} > \frac{1}{n}$

3. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E . Prove that $\alpha \leq \beta$.

Proof. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E . Since α is a lower bound of E , $\alpha \leq x \forall x \in E$. Similarly, since β is an upper bound for E , $\beta \geq x \forall x \in E$, i.e. $x \leq \beta \forall x \in E$. By transitivity of the relation, $\alpha \leq x \leq \beta \forall x \in E$. Therefore, $\alpha \leq \beta$. □

4. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$ where $x \in A$. Prove that $\inf(A) = -\sup(-A)$.

Proof. Let A be a nonempty set of real numbers which is bounded below. By the problem below, A has a greatest lower bound, say $\inf A = \alpha$. Since $\alpha = \inf A$, if β is any other lower bound, $\beta \leq \alpha$.

Let $-A$ be the set of all numbers $-x$ where $x \in A$, i.e. $-A = \{-x \mid x \in A\}$. Then we want to show that $-A$ has a supremum, which is equal to $-\inf A$.

If β is any lower bound for A , $\beta \leq x \forall x \in A$, so that $-\beta \geq -x \forall x \in A$. This means that $-\beta$ is an upper bound for $-A$. In particular, $-\alpha$ is an upper bound for $-A$, and since $\beta \leq \alpha$ for all lower bounds β , $-\alpha \leq -\beta$. It remains to prove that there are no other upper bounds of $-A$ which are smaller than $-\alpha$. Assume that γ is an upper bound for $-A$ and that $\gamma < -\alpha$. Then $\alpha > -\gamma$, and since γ is an upper bound for $-A$, $-x \leq \gamma \forall -x \in -A$. Therefore $x \geq -\gamma \forall x \in A$. Therefore $-\gamma$ is a lower bound of A and since $\alpha = \inf A$, $\alpha \geq -\gamma$, a contradiction. Therefore $-\alpha = \sup(-A)$. □

In Class. Suppose S is an ordered set with the lub-property, $B \subset S$, $B \neq \emptyset$ and B is bounded below. Let L be the set of all lower bounds of B . Then $\alpha = \sup L$ exists in S and $\alpha = \inf B$. In particular, $\inf B$ exists in S .

Proof. Assume that S is an ordered set with the lub-property. Let $B \subset S$ with $B \neq \emptyset$, and let B be bounded below. Let L be the set of all lower bounds of B . Then L is a subset of S , and therefore has the lub-property. L is bounded above by any of the elements of B , and so L has a least upper bound in S , call it α . Then by definition, $\alpha = \inf B$ since it is the supremum of the set of lower bounds of B . \square

5. Let $\beta \in \mathbb{R}$, $p, q, p_1, q_1 \in \mathbb{R}^n$. Then

$$(\beta p) \cdot q = \beta(p \cdot q) = \beta(q \cdot p)$$

and

$$p \cdot (q_1 + q_2) = p \cdot q_1 + p \cdot q_2$$

6. You should be able to do problems 6-10, 12-13 in the Exercises on pages 10-11. I will not collect these.

7. You should also be able to do the following problem, which I will not collect: Find the sup and inf of each of the following sets of real numbers:

(a) All real numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$ where $p, q, r \in \mathbb{Z}^+$.

(b) $S = \{x \mid 3x^2 - 10x + 3 < 0\}$

(c) $S = \{x \mid (x - a)(x - b)(x - c)(x - d) < 0\}$ where $a < b < c < d$.