Math 560
Fall, 2005 Name: **Solutions**
Exam 1

Show all work and explain your reasoning. Answer all questions. Good Luck!!!

1. **Definitions.** (2 points each = 6 points) Complete each statement with the required definition.

   (a) A sequence $\{p_n\} \subset \mathbb{R}^n$ converges to $p \in R^n$ if
   for every $\varepsilon > 0$ there is a number $N > 0$ so that $| p_n - p |< \varepsilon$ whenever $n > N$.

   (b) A function f from a domain $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$ is uniformly continuous on D if
   for every $\varepsilon > 0$ there is a $\delta > 0$ so that $| f(p) - f(q) |< \varepsilon$ whenever $| p - q |< \delta$ for $p, q \in D$.

   (c) A set X is connected if
   there do not exist two disjoint open sets A and B, both of which are non-empty, so that $A \cup B = X$.

2. **Theorems.** (3 points each = 12 points) State the following theorems:

   (a) The Cauchy-Schwartz Inequality
   Let $p, q \in \mathbb{R}^n$. Then $p \cdot q \leq | p || q |$.

   (b) The Nested Interval Theorem
   Let $\{I_n\}$ be a sequence of nonempty bounded closed intervals on the line which are montonic decreasing (i.e. nested) in the sense that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Then $\bigcap_{i=1}^{\infty} I_n \neq \emptyset$, so that there must exist at least one point p that lies in all of the I_n. Moreover, if the length of I_n approaches 0 then the intersection is exactly one point.

   (c) The Bolzano-Weierstrass Theorem for $\mathbb{R}^n$
   Every bounded infinite set of real numbers has a cluster (limit or accumulation) point. Every bounded infinite set in $\mathbb{R}^n$ has a cluster point. Moreover, any bounded sequence in $\mathbb{R}^n$ has a limit point, and thus a convergent subsequence.

   (d) The Heine-Borel Theorem
   A subset S of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

3. **True / False.** State if the following are true or false. If true, provide a brief proof. If false, provide a counterexample.

   (a) (4 points) The set $\mathbb{Q}$ is an ordered field.
   TRUE. The easiest way to say this is that since $\mathbb{Q}$ is a subfield of the ordered field $\mathbb{R}$, $\mathbb{Q}$ is also an ordered field.

   (b) (4 points) If $E \subset \mathbb{R}$, the interior of E and the interior of the closure of E are the same.
   FALSE. There are lots of good examples. For instance, let $E = \mathbb{R} - \{1\}$. Then E is open, so it is its own interior. Then $\overline{E} = \mathbb{R}$, which is clopen. Therefore, $int\mathbb{R} = \mathbb{R}$, which is different from $int(E)$.

(c) (4 points) There is an open cover of $(0,1)$ which has no finite subcover.

TRUE. An easy example is $\left(\frac{1}{n}, 1-\frac{1}{n}\right)$ for $n \in \mathbb{N}$. This is an open cover of $(0,1)$, but it has no finite subcover.

(d) (6 points) If $\{p_n\} \subset \mathbb{R}^n$ is a sequence that converges to $p_0 \in \mathbb{R}^n$, then $\{p_n\} \cup \{p_0\}$ is a compact set.

TRUE. This can be proven by Heine-Borel. However, there is a simple direct proof.

Since $p_n \to p_0$ we know that for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ so that $\mid p_n - p_0 \mid < \varepsilon$ whenever $n \geq N$. Therefore, there are only finitely many terms $p_1, \ldots, p_{N-1}$ which are not within ε of p_0. Put an open ball around each of them. Therefore, you can always find a finite subcover of any infinite cover.

(e) (7 points) There exists an unbounded continuous function on $\mathbb{R}$ which is not uniformly continuous.

TRUE. Let $y = x$. This function is continuous, unbounded, and not uniformly continuous.

(f) (7 points) A continuous function maps an open set to an open set.

FALSE since $f(x, y) = x^2$ maps $\mathbb{R}$ to $[0, \infty)$.

Prove or disprove each of the following. Be sure that your explanations are clear and complete.

4. (10 points) Prove that S in n space is compact if and only if every sequence in S has a limit point that belongs to S.

Proof. Suppose that S is compact. Then it is closed and bounded. Take a sequence $\{p_n\} \subset S$. Since $\{p_n\}$ is bounded, there is a convergent subsequence. So a subsequence of $\{p_n\}$ converges to some $p_0 \in \mathbb{R}^n$. Since S is closed, $p_0 \in S$.

Suppose every sequence in S has a subsequence that converges to a point in S. Then it would suffice to show that S is closed and bounded (by the Heine-Borel Theorem). If S were unbounded, then S would contain a sequence $\{p_n\}$ such that $\mid p_n - 0 \mid = \infty$, and then no subsequence would converge. Thus S is bounded. If S were not closed then there would be a sequence of $\{p_n\} \subset S$ that would converge to $p_0 \notin S$. Since then every subsequence would converge to $p_0 \notin S$, we have a contradiction. Therefore, S is closed and bounded, hence compact. □

5. (10 points) Let A and B be disjoint closed sets and suppose f is uniformly continuous on each. Show that f is uniformly continuous on $A \cup B$ if A is compact.

Proof. Since A is compact, the distance between the sets A and B is positive. Therefore, if $p, q \in A \cup B$ and $\mid p - q \mid < \delta$ then both $p, q \in A$ or $\mathcal{P}, q \in B$. Then they satisfy the definition of uniformly continuous on A or on B.

Notice that if we have $A = \{(x, y) \in \mathbb{R}^2 : y \geq 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : y \leq \frac{x}{x+2}\}$ then the sets are both closed, but unbounded. Since they get arbitrarily close, you don't get to say that $p, q \in A$ or $p, q \in B$. Therefore, you could have $p \in A$ and $q \in B$, which would not allow you to use the fact that f was uniformly continuous on each set. However, if at least one of the sets is also bounded, then you can find a positive distance between them. □

6. (10 points) Prove that every convergent sequence in $\mathbb{R}^n$ is Cauchy.

Proof. Assume that $\{p_n\}$ is a sequence in $\mathbb{R}^n$ converging to p. Then let $\varepsilon > 0$ be given. Then $\frac{\varepsilon}{2} > 0$ also. Since $\{p_n\}$ converges, we know that there exists an $N \in \mathbb{N}$ so that $\mid p_n - p \mid < \frac{\varepsilon}{2}$ and $\mid p_m - p \mid < \frac{\varepsilon}{2}$ when $n, m > N$.

Consider $n, m > N$ and look at

$$\begin{aligned} \mid p_n - p_m \mid &= \mid p_n - p + p - p_m \mid \\ &\leq \mid p_n - p \mid + \mid p - p_m \mid \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

making the sequence Cauchy. □

7. (10 points) If f is defined on E, the graph G of f is the set of points $(x, f(x))$ for $x \in E$. In particular. if E is a set of real numbers and f is real valued, the graph of f is a subset of the plane. Suppose E is compact and prove that f is continuous on E if and only if its graph G is compact.

Proof. Assume E is compact and f is continuous on E. Then the graph G is

$$E \times f(E) = \{(x, f(x)) : x \in E, f(x) \in f(E)\}$$

Any open cover of $E \times f(E)$ is a set $W_\alpha \times U_\alpha$ where U_α is an open cover of E and V_α is an open cover of $f(E)$. Since E is compact, and f is continuous, $f(E)$ is compact. Also, since E is compact, there is a finite subcover $U_1, \ldots, U_n$, and since $f(E)$ is compact, there is a finite subcover $V_1, \ldots, V_m$. Therefore, there is a finite subcover $U_1 \times V_1, \ldots U_n \times V_m$ of $E \times f(E)$. Therefore the graph G is compact.

To prove the converse, assume that E is compact, G is compact, and f is not continuous. Then for every $\delta > 0$ there is an x so that $\mid x - x_0 \mid < \delta$ but $\mid f(x) - f(x_0) \mid > \varepsilon$. Choose $\{x_n\} \subset E$ with $\mid x_n - x_0 \mid < \frac{1}{n}$ and $\mid f(x_n) - f(x_0) \mid > \varepsilon$. Then $x_n \to x_0$ since $\mid x_n - x_0 \mid < \frac{1}{n}$. Then $\{(x_n, f(x_n))\}$ is a sequence in G. Since G is compact, it is closed and bounded, so there is a subsequence $\{(x_{n_k}, f(x_{n_k}))\}$ which converges. Since $x_n \to x_0$, we know that the every subsequence converges to the same point, i.e. $x_{n_k} \to x_0$. But since we assume that f is not continuous, $f(x_{n_k}) \not\to f(x_0)$. But since G is compact, for every sequence in G there is a subsequence which converges so $\{(x_{n_k}, f(x_{n_k}))\} \to (x_0, f(x_0))$, a contradiction.

Therefore f is continuous. □

8. (10 points) Suppose $f : [0,1] \to [0,1]$ is continuous. Prove that there exists at least one $x \in [0,1]$ for which $f(x) = x$.

   Proof. **(Sieradski, p. 88)** If either $f(0) = 0$ or $f(1) = 1$ we are done.

   Otherwise $f(0) > 0$ and $f(1) < 1$. Then let $h(x) = f(x) - x$. Notice that h is also continuous (since it is the difference of continuous functions). Then $h(1) = f(1) - 1 < 0$ since $f(1) < 1$ and $h(0) = f(0) - 0 > 0$ since $f(0) > 0$. Therefore, by the intermediate value theorem, there is a place where $h(x) = 0$, i.e. $f(x) = x$. □