Math 477 - Homework
Fall 2006

Problems about quotient groups.

Definition 7 *Let n be a positive integer. Look at the set $\mathbb{Z}_n = \{[0],[1],[2],\ldots,[n-1]\}$where $[a] = \{$ integers congruent to $a(modn)\}$, i.e. two numbers are in the same equivalence class if they have the same remainder when divided by n. We can define an operation on $\mathbb{Z}_n$ by $[a]+[b] = [a+b]$.*

68. Prove that this operation on $\mathbb{Z}_n$ is well-defined. That is, show that if $a, b \in [a]$ and $c, d \in [c]$ then $[a]+[b] = [c]+[d]$.
69. Prove that $\mathbb{Z}_n$ is a group under addition $(modn)$.

Definition 8 *Let G be a group and $N \triangleleft G$. Define an operation on $\{Na \mid a \in G\}$ (the set of right cosets of N in G) by $(Na)(Nb) = N(ab)$.*

70. Prove that this operation is well-defined.
71. Explain why $N \triangleleft G$ is necessary for this operation to be well-defined.

Definition 9 *Let G be a group and $N \triangleleft G$. Let $G/N = \{Na \mid a \in G\}$ be the set of right cosets of the normal subgroup N in G. Define $(Na)(Nb) = N(ab)$ as above.*

72. Prove that G/N is a group under the multiplication defined above.
73. Prove Cauchy's Theorem: If G is a finite abelian group of order $\mid G \mid$ and p is a prime that divides $\mid G \mid$, then G has an element of order p.
74. If G is a group and $N \triangleleft G$, show that if $\overline{M}$ is a subgroup of G/N and $M = \{a \in G \mid Na \in \overline{M}\}$ then $M \leq G$ and $N \subset M$.
75. Let G be a cyclic group and N is a subgroup of G. Prove or disprove: G/N is a cyclic group. (Question: Why does G/N make sense?)
76. Let G be an abelian group and N is a subgroup of G. Prove or disprove: G/N is an abelian group. (Question: Why does G/N make sense?)
77. Let G be a group. Prove or disprove: If $G/Z(G)$ is cyclic then G is abelian.
78. Let G be a group and $N \triangleleft G$. Then G/N is abelian iff $aba^{-1}b^{-1} \in N$ for all $a, b \in G$.
79. Let G be an abelian group (possibly infinite) and let $T = \{a \in G \mid a^m = e, m > 1$ depending on $a\}$.
    (a) Prove or disprove: $T \leq G$
    (b) Prove or disprove: G/T has no element (other than its identity) of finite order.