The Existence of Infinity through Transfinite Numbers

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Abstract

In this paper, we will discuss the evolution of thought towards the idea of infinity. We will then examine Georg Cantor's theory of transfinite numbers while simultaneously disproving the mathematical opinions relating to infinity historically held by some of the world's prominent philosophers.

1 Introduction

The concept of an absolute infinite is a vast subject to comprehend. In this paper, we will begin at the very beginning with proof of its actual existence through Georg Cantor's theory of transfinite numbers, which contradicted many great philosophers' ideals upon the denial of an absolute infinite existing in any aspect of the universe. We will view in detail the opinions of a few of history's notable philosophers upon the subject of infinity and briefly discuss Cantor's research and his development of his threefold division of the Infinite. We will then introduce the definition of transfinite numbers and the procedure for creating ordinal numbers to prove the existence of an absolute infinite.

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2 The Philosophers

The great philosophers of our history could not or would not grasp the idea of actual infinite thing existing in the universe in which they lived. Greek philosopher and mathematician Pythagoras(580 B.C. - 500 B.C.) believed that anything in the world could be represented as a finite arrangement of natural numbers. By natural numbers Pythagoras actually meant whole or counting numbers. Greek philosopher Plato (427 - 347 B.C.) believed that even his ultimate form of the universe, the Good, was finite and definite. This idea contradicted the beliefs of the metaphysicians at the time who believed that the highest form was indeed infinite. Greek philosopher and scientist Aristotle (384 -322 B.C.) believed that there were many aspects of the finite world that ”appeared” to be infinite but were actually definite. In other words, he believed that certain things, such as time, had the potential to be actually infinite. The Egyptian-Roman philosopher Plotinus (205 - 207 A.D.) accepted what Plato could not and agreed with the beliefs of metaphysicians at the time that at least God, or the One, was indeed infinite.

Christian theologian St. Augustine (354 - 430 A.D.) went one step above the idea of the One, or the higher power being infinite and added the idea that The One could think infinite thoughts. Philosopher and theologian of the Roman Catholic Church Saint Thomas Aquinas (1225 - 1274 A.D.) believed that even though God's power is infinite he cannot make an absolutely infinite thing, just as he could not make and ”unmade” thing because that would mean to things that contradicted each other would exist and be true at the same time [4]. Aquinas was one of the only philosophers that came up with a substantial argument to the nonexistence of an infinite set. He stated that there can only be an infinite set if infinite numbers existed, which he believed did not.

These philosophers, along with others could accept that the highest power of their universe (i.e. God, the Good, the One) was actually infinite and unbounded, but could not grasp the concept that anything else could be. Aquinas' opinion about an infinite set was proven wrong by the later research of Georg Cantor and his theory of transfinite numbers.

3 Georg Cantor

Georg Ferdinand Ludwig Phillip Cantor (1845 - 1918) was a German mathematician who was the creator of set theory, the mathematics of the infinite, in 1873.[3] Through his research and theory of transfinite numbers, he was able to show that there were different degrees of infinity. He kept the idea of the Absolute Infinite but formed intermediate levels between the finite and absolute which consist of physical infinities, mathematical infinities, and the Absolute Infinite.

3.1 Threefold Division of the Infinite

Physical Infinities consist of the infinities that exist in the world around us. Is space infinite? Our universe does appear to be infinite but is it really? Could there be a possibility that our universe is bounded by some spacial sphere? But what about the space beyond that boundary? Space does seem to be infinite and unbounded. What about time? The subject of time has been the center of many arguments and theories by metaphysicians and scientists. Our world and how we measure time have started with the Big Bang theory. Will there be a Big Stop? Or is time circular with many beginnings after the many endings? The theory of matter also falls into this category. Take, for example a rock. A rock is actually a collection of molecules that make up the rock. That collection of molecules is made up of a collection of atoms, which are a collection of particles which are a collection of quarks. By induction we would assume that quarks would have to be a collection of something else.

An ultimate body over the universe was the only aspect of the Absolute Infinite the medieval philosophers could accept. Given many names throughout the ages, the higher power consists of an absolute being called the One, the Good, or God. This level of the infinite contains very interesting and somewhat complex ideas. The Mindscape, which is the infinity that dwells in our minds, is considered a part of the Absolute Infinite. A person could be thinking of a thought consisting of a collection of different things. If that person thinks long enough, he or she would come across the thought of thinking about that collection of things. Now their complete thought consists of a thought consisting of a collection of things. Thinking and analyzing our own thoughts is a completely unbounded action: an infinity within the mind.

The idea of infinity is used largely in the study of calculus. In integration, we use an infinitesimally small integral to best calculate the area beneath a curve. Infinity is also expressed in the calculation of limits of certain function. The fathers of calculus (such as Descartes, Fermat, and Newton) did not use the actual infinite in there study of the "limit" but instead focused on a function's "becoming" infinite or rather being potentially infinite.[1] One aspect of mathematical infinities will be our discussion on transfinite number.

4 The Numbers

Transfinite numbers are numbers that are infinite but conceivable. They are often called ordinal numbers. Ordinal numbers indicate the order of elements in a set or series such as first or second. Cardinal numbers are the numbers that express the quantity of elements in a set. In this paper we will focus of the ordinal numbers. There are two principles for generating ordinal numbers.

1. Principle I Given any ordinal number a, then the next ordinal number can be found, called $a+1$ [4].

2. Principle II If given some particular sequence of increasing ordinals a, the last ordinal of all a's is called the $\lim a$, [4].

5 The Steps

To begin our production of ordinal numbers we first begin with our first ordinal number 0. The reason being that when we start with a set, we would naturally begin with the set that contains nothing within it. We apply our first principle repeatedly to create a sequence of ordinals

$$0, 1, 2, 3, \ldots, n$$

We then apply the second principle to continue further beyond our sequence we have $\lim n = \omega$. We now have the sequence

$$0, 1, 2, 3, \ldots, \omega.$$

ω is the first infinite ordinal number because it is larger than the ordinals generated before it. We can then apply the first principle to continue our ordinal numbers.

$$0, 1, 2, 3, \ldots, \omega, \omega+1, \omega+2, \omega+3, \ldots \omega+n.$$

$\omega+1$ does not actually mean the sum of ω and 1. We have just picked a symbol that we can identify as the ordinal number after ω.[2] Using the second principle we have

$$\lim(\omega+n) = \omega+\omega = \omega \cdot 2.$$

After using the second principle to go beyond our definite sequence of ordinals, we now have the sequence

$$0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega\cdot 2.$$

Repeating the first principle on our current sequence of ordinals we now have

$$0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega\cdot 2, (\omega\cdot 2)+1, (\omega\cdot 2)+2, \ldots, (\omega\cdot 2)+n.$$

Taking the limit of $(\omega\cdot 2)+n$, we get $\omega\cdot 3$, which produces the sequence

$$0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega\cdot 2, (\omega\cdot 2)+1, (\omega\cdot 2)+2, \ldots, \omega\cdot 3.$$

By continuing to use Principle I on our current sequence we then come to $\omega\cdot n$ for each ordinal number n. Apply the second principle yields the $\lim(\omega \cdot n) = \omega \cdot \omega = \omega^2$. This limit produces the sequence:

$$0, 1, 2, 3, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega\cdot 2, \omega\cdot 3, \omega\cdot 4, \ldots, \omega^2, \omega^3, \ldots, \omega^\omega, \ldots$$

5.0.1 Tetration

Tetration is the next logical step in iteration after exponentiation. The prefix tetra- which means 4 (forth step: addition, multiplication, exponentiation, tetration) and the word iteration which means the act of repeating. Mathematically, it is the act of repeatedly applying a function upon itself. The reason why tetration is not used as often in everyday mathematics as other functions is because even small numbers yield large products. Below we use the iteration on the number three as an example.

addition: $3 + 3 = 6$
multiplication: $3 \cdot 3 = 9$
exponentiation: $3^3 = 27$
tetration: ${}^{3}3 = 3^{3^3} = 19{,}683$

To continue further in our production of ordinal numbers, we have to apply the function of tetration. After applying our first and second principles, we can continue our sequence as follows:

$$0, 1, 2, 3, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega\cdot 2, \omega\cdot 3, \omega\cdot 4, \ldots, \omega^2, \omega^3, \ldots, \omega^\omega, \ldots {}^{\omega}\omega \ldots$$

6 Conclusion

This leads us to the Georg Cantor's Absolute Infinity, Ω. Ω is not a definite ending to the sequence of ordinal numbers because it contains every and all possible ordinal numbers leading up to Ω. We can refer to Ω as our first uncountable ordinal number.[**?**] By induction, we could continue to produce uncountable ordinals beyond Ω and onto $\Omega+1$, $\Omega\cdot 2$, Ω^2, Ω^Ω, and onto another set of uncountable ordinals.

The mathematical opinions and objections to the thought of anything progressing to an actual infinite was proven wrong by Georg Cantor's theory of transfinite numbers and the proof of there existing infinitely many ordinal numbers. Although only proving an infinity in transfinite number, the theory proved that an actual infinity does exist.

References

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