HOMEOMORPHISMS, DIFFEOMORPHISMS AND PRIME NUMBERS

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Abstract. Using a construction of Kreck and Stolz, we find a 2-parameter family $\{W_{k,l}\}_{k,l\in\mathbb{Z}}$ of real 8-dimensional smooth orientable manifolds with homeomorphic but not necessarily diffeomorphic boundaries. Adjusting these boundaries by the connected sum with a particular number of 8-dimensional punctured Bott manifolds, the boundaries become diffeomorphic (up to attaching a cylinder). We may then attach these manifolds at the boundary, forming a closed 8-dimensional manifold X. The $\hat{A}$-genera of these manifolds $X_{k,l}$ are interesting in the fact that they appear to produce very large prime numbers. We examine the distribution and characterization of these primes.

1. Introduction

What if we could develop a system to generate all the possible prime numbers with the aid of just two special integers. In our research we tried to develop this system and understand if and how we can do this with the help of a concept presented in a paper by Kreck and Stolz. To get a grasp on this concept we must define a few topological terms.

2. Definitions and Examples

2.1. What is a topological space.

Definition 1. *A topological space is a set E along with an assignment to each $p \in E$ of a collection of subsets of E, to be called neighborhoods of p, and satisfying the for properties;*

1. *p belongs to any neighborhood of p.*
2. *If U is a neighborhood of p and $V \supset U$, then V is a neighborhood of p.*
3. *If U and V are neighborhoods of p, so is $U \cap V$.*
4. *If U is a neighborhood of p, then there is a neighborhood V of p such that $V \subset U$ and V is a neighborhood of each of its points.*

2.2. **What is a homeomorphism.**

Definition 2. *Let f be a one-to-one map of E onto F, where E and F are topological spaces. Thus there is an inverse map g of F onto E. If both f and its inverse are continuous, f will be called a homeomorphism and E and F will be said to be homeomorphic.*

In looking at a homeomorphism, the key property is that the map and its inverse are continuous and one-to-one. A homeomorphism can simply be stated that if two topological spaces are homeomorphic to each other then that tells us that they are the same. Let us look at a basic circle and some manipulations of the circle , and show an example of a homeomorphism.

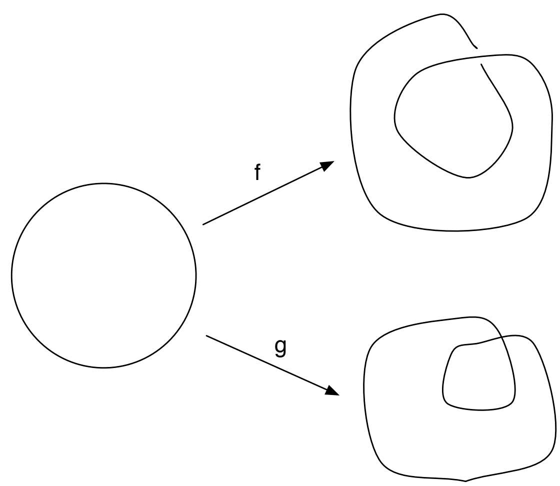

FIGURE 1. Map f is a homeomorphism and Map g is not a homeomorphism

Notice that in Figure 1 if we pick any point on the circle we can see that f is a map that is homeomorphic to the circle that has a slight twist, but the strands of the twisted circle never touch at the crossing. In picking a point on the circle we look at the surrounding points around the selected point, notice that on the circle and the twisted circle the neighborhood of point are the same, which makes them homeomorphic to each other. Now look at the circle with the map g to the circle that is twisted, but at the crossing that has been fused together. Just like before we look at the neighborhood of points around the selected point. Notice that on the fused twisted circle at the neighborhood has four strands entering, while on the circle every neighborhood has only two strands entering. Therefore these two shapes are not homeomorphic.

2.3. **What is a differentiable manifold.**

To understand what a manifold is we must look at what it means for a topological space to be Hausdorff.

Definition 3. *A topological space will be called Hausdorff if, for any two distinct points p and q, there are neighborhoods U of p and V of q such that $U \cap V = \emptyset$. Thus distinct points are separated by disjoint neighborhoods.*

The clearest way to think about this as well as remember it is think of Hausdorff as Housed-off. What is meant by that is each point has a neighborhood of Euclidean space that surrounds it and keeps it Housed-off from other point so that the two spaces do not have any points in common with one another, which makes the neighborhoods disjoint.

Definition 4. *An n-dimensional differentiable manifold M is a Hausdorff topological space that has a covering by countably many open sets $U_1, U_2, \ldots,$ satisfying the following conditions:*

- *for each U_i there is a homeomorphism $\phi_i : U_i \to V_i$ where V_i is an open cell in Euclidean n-space.*
- *If $U_i \cap U_j \neq \emptyset$, the homeomorphisms ϕ_i and ϕ_j compose to give a homeomorphism $\phi_{ji} = \phi_j \circ \phi_i^{-1}$ of $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j)$, which is a differentiable map.*

In examining a manifold the simplest way to understand the definition is in a picture.

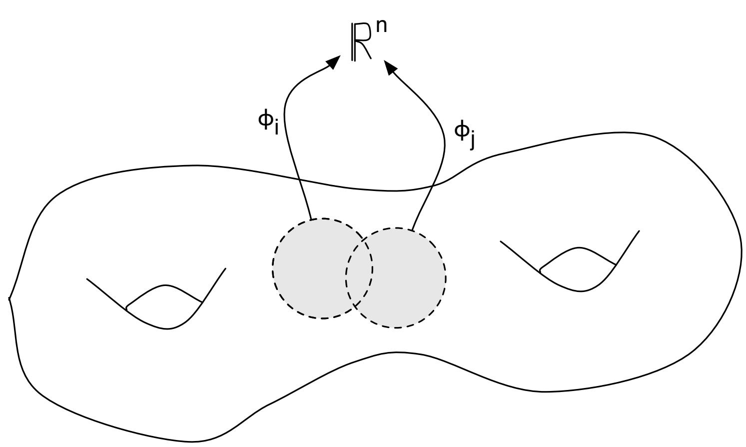

FIGURE 2. Example of a manifold

Looking at Figure 2 we can better understand what a manifold is. Each point on the surface of the manifold has a neighborhood of Euclidean space and those neighborhoods intersect each other. At these intersections we could have a possible problem, but by the second part of the definition we show how the intersection is taken care of. Notice how the functions ϕ_i and ϕ_j map points on the surface to $\mathbb{R}^n$, and by using the inverses of the functions we can see how the intersections does not cause a problem.

2.4. **What is a diffeomorphism.**

Definition 5. *Let M and N be differentiable manifolds, and let f be a one-to-one differentiable map of M onto N such that the inverse map is also differentiable. Then f is called a diffeomorphim and the manifolds M and N are said to be diffeomorphic.*

A simple way to look at a diffeomorphism is that it is a homeomorphism with a special property. The property being that the map that is homeomorphic for two spaces can be differentiated. Here are some examples of what it means for two spaces to be diffeomorphic, and homeomorphic but not diffeomorphic.

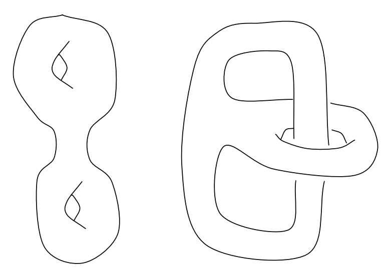

FIGURE 3. Example of a diffeomorphic two handle manifold

Notice in Figure 3 that the two handle manifold has just been cut and looped through the other handle then pieced back together.

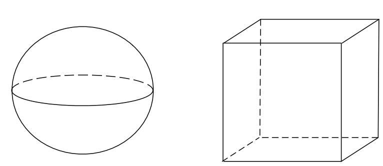

FIGURE 4. Sphere not diffeomorphic Cube

Look at the sphere and the cube in Figure 4, notice that we can create a one-to-one map with and inverse mapping every point on the sphere to the cube. We can all see that the corners on the cube are similar to the apex of the Absolute value function, and we know through calculus that at the apex of the absolute value function there does not exist a derivative. Therefore the sphere and the cube are not diffeomorphic to each other.

2.5. **What is a manifold with boundary.**

Definition 6. *A differentiable manifold of dimension n with boundary is a topological space M with a subspace N and a countable open covering $U_1, U_2, \ldots,$ with homeomorphisms $\phi_1, \phi_2, \ldots,$ satisfying the following conditions:*

- *Each set U_i of the given covering is either contained in $M - N$, in which case there is a homeomorphism $\phi_i : U_i \to V_i$, where V_i is a solid open sphere in n-space, or otherwise there is a homeomorphism $\phi_i : U_i \to V_i$, where V_i is a hemisphere of the form $\sum_1^n x_i^2 < 1, x_n \geq 0$, the set $U_i \cap N$ being mapped on the subset of V_i for which $x_n = 0$*
- *If U_i and U_j are two sets of the given covering and if ϕ_i and ϕ_j are the homeomorphisms just described and if $U_i \cap U_j \neq \phi$, then $\phi_{ij} = \phi_i \phi_j$ is a differentiable map of $\phi_j(U_i \cap U_j)$ onto $\phi_i(U_i \cap U_j)$.*

This is better explained in a picture.

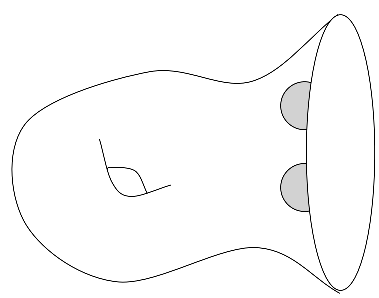

FIGURE 5. Manifold with Boundary

Each point on the surface of the manifold has a neighborhood of Euclidean space but if the manifold has boundary the points on the boundary only have a neighborhood of half of Euclidean space. Imagine if you will an ant walking on a huge surface where we see its curvature, but the ant feels like it is on a plane. What if the surface had a hole somewhere in it and the ant approached any of the points creating the hole. Say the hole lead to some Great Abyss, so that when the ant walks around the hole it felt like walking a line, where if it took one wrong step it either fell in the Abyss or slid down the surface. That line which creates the neighborhood of half of Euclidean space would be considered a boundary for that manifold.

3. Kreck and Stolz

Using the standard metric on $\mathbb{C}P^1 \times \mathbb{C}P^2$, let $M_{k,l}$ be the total space of the S^1-bundle characterized by $c_1 = lx + ky$, where x, y are generators of $H^2(\mathbb{C}P^2)$ and $H^2(\mathbb{C}P^1)$, respectively. Let $W_{k,l}$ be the total space of the D^2-bundle bounding $M_{k,l}$.

If k is even, then $M_{k,l}$ and $W_{k,l}$ are Spin 8-dimensional manifolds, and if $(k,l) = 1$, then l is odd, so $\pi_1(M_{k,l}) = 0$.

Theorem (Kreck/Stolz) For even k, $(k,l) = 1$, there is $a(l) \in \mathbb{N}$, such that if $l' = \pm l$ and $k' \equiv k \pmod{a(l)}$ then $M_{k',l'}$ is diffeomorphic to $M_{k,l}$.

In this same paper, it is proved that for each k, l, there is $s(k,l) \in \mathbb{Q}$ such that,

$$k \neq k' \Leftrightarrow s(k,l) \neq s(k',l')$$

In this case, the metrics associated to $M_{k,l}$ and $M_{k',l'}$ are not in the same path component of $Riem^+(\mathrm{Diff}(M_{k,l}))$.

But, is there $\Phi \in \mathbb{Z}$ so that $s(k,l) + \Phi = s(k',l')$?

If so, then we can form the manifold $(W_{k,l} \# \Phi \cdot B^8) \cup W_{k',l'}$ a closed 8-dimensional Bott manifold.

4. Our Problem

4.1. What if $\Phi \in \mathbb{Z}$.

We start with k and $l \in \mathbb{Z}$, so that k is even and that the $gcd(k,l) = 1$.

Notation:

Define

$$\lambda_2 \equiv \begin{cases} 0 & \text{if } l \equiv 2,6 \\ 1 & \text{if } l \equiv 1,7 \\ 2 & \text{if } l \equiv 3,5 \\ 3 & \text{if } l \equiv 0,4 \end{cases}$$

The λ_2 depends on the value of l modulo 2.

$$\lambda_7 \equiv \begin{cases} 0 & \text{if } l \equiv 1,2,5,6 \\ 1 & \text{if } l \equiv 0,3,4 \end{cases}$$

The λ_7 depends on the value of l modulo 7.

Note that for every value of k and l there will be a different λ_2 and λ_7

Define

$$a(l) \equiv 2^{\lambda_2} \cdot 7^{\lambda_7} \cdot l^2$$

, which depends on both the λ_2 and λ_7, so this is also a function of l.

Let

$$l' = \pm l \text{ and } k' = k + m \cdot [a(l)]$$

The next two pieces of notation are taken form the paper by Kreck and Stolz, which allow us to find a rational number to represent any two manifolds with boundary and weather or not they are diffeomorphic to each other.

Define

$$s(k,l) \equiv \frac{-3k(l^2+3)(l^2-1)}{2^7 \cdot 7 \cdot l^2}$$

Define

$$s(k',l') \equiv \frac{-3[k+m \cdot a(l)](l^2+3)(l^2-1)}{2^7 \cdot 7 \cdot l^2}$$

Now we can ask what must be true about Φ if

$s(k,l) + \Phi = s(k',l')$?

First we substitute for each term and expand out. Notice that since l and l' are always squared that the fact $l' = \pm l$ does not play a factor because that value will always be positive.

$$\frac{-3k(l^2+3)(l^2-1)}{2^7 \cdot 7 \cdot l^2} + \Phi = \frac{-3[k+m \cdot a(l)](l^2+3)(l^2-1)}{2^7 \cdot 7 \cdot l^2}$$

Solving for Φ, and simplifying we get:

$$\Phi = \frac{3k(l^2+3)(l^2-1) - 3[k+m \cdot a(l)](l^2+3)(l^2-1)}{2^7 \cdot 7 \cdot l^2}$$

$$\Phi = \frac{(l^2+3)(l^2-1)(3k-3k+3m \cdot a(l))}{2^7 \cdot 7 \cdot l^2}$$

$$\Phi = \frac{-3m \cdot a(l)(l^2+3)(l^2-1)}{2^7 \cdot 7 \cdot l^2}$$

We now have an expression for Φ, so let us look at its prime factorization and see if there is anything special about the primes it gives us. First we need to try and single out the largest prime factor of Φ.

5. New Idea: Using the Quadratic Equation

Recall that,

$$a(l) \equiv 2^{\lambda_2} \cdot 7^{\lambda_7} \cdot l^2$$

. Then

$$\Phi = \frac{-3m \cdot a(l)(l^2+3)(l^2-1)}{2^7 \cdot 7 \cdot l^2}$$

$$\Phi = \frac{-3m(2^{\lambda_2} \cdot 7^{\lambda_7} \cdot l^2)(l^2+3)(l^2-1)}{2^7 \cdot 7 \cdot l^2}$$

$$\frac{\Phi(2^7 \cdot 7)}{-3m(2^{\lambda_2} \cdot 7^{\lambda_7})} = (l^2+3)(l^2-1)$$

Substitute $w = l^2$

$$\frac{\Phi(2^7 \cdot 7)}{-3m(2^{\lambda_2} \cdot 7^{\lambda_7})} = (w+3)(w-1)$$

$$w^2 + 2w - 3 + \frac{\Phi(2^7 \cdot 7)}{3m(2^{\lambda_2} \cdot 7^{\lambda_7})} = 0$$

$$w = \frac{-2 \pm \sqrt{4 - 4(-3 + \frac{\Phi(2^7 \cdot 7)}{3m(2^{\lambda_2} \cdot 7^{\lambda_7})}}}{2}$$

$$w = -1 \pm \sqrt{4 - \frac{\Phi(2^7 \cdot 7)}{3m(2^{\lambda_2} \cdot 7^{\lambda_7})}}$$

Therefore,

$$l = \sqrt{-1 \pm \sqrt{4 - \frac{\Phi(2^7 \cdot 7)}{3m(2^{\lambda_2} \cdot 7^{\lambda_7})}}}$$

This expression for w allowed us to create way of finding what integer l had to be for a given prime factor of Φ. Lets look at what exactly would divide into Φ to possibly prove that any prime number will divide into Φ give the two integers k and l.

6. Conclusion

6.1. **Proof that any Prime will divide Φ.** Again recall that:

$$a(l) \equiv 2^{\lambda_2} \cdot 7^{\lambda_7} \cdot l^2$$

.

and

$$\Phi = \frac{-3m \cdot a(l)(l^2+3)(l^2-1)}{2^7 \cdot 7 \cdot l^2}$$

Since we know that by the premises that the k is even and $gcd(k, l) = 1$ we can infer that l must be odd. So let $l = 2d + 1$

$$\begin{aligned}
\Phi &= \frac{-3m \cdot a(l)((2d+1)^2+3)((2d+1)^2-1)}{2^7 \cdot 7 \cdot l^2} \\
&= \frac{-3m((2d+1)^2+3)((2d+1)^2-1)(2^{\lambda_2} \cdot 7^{\lambda_7} \cdot l^2)}{2^7 \cdot 7 \cdot l^2} \\
&= \frac{-3m((2d+1)^2+3)((2d+1)^2-1)(2^{\lambda_2} \cdot 7^{\lambda_7})}{2^7 \cdot 7} \\
&= \frac{-3m((4d^2+4d+1)+3)((4d^2+4d+1)-1)(2^{\lambda_2} \cdot 7^{\lambda_7})}{2^7 \cdot 7} \\
&= \frac{-3m(4d^2+4d+4)(4d^2+4d)(2^{\lambda_2} \cdot 7^{\lambda_7})}{2^7 \cdot 7} \\
&= \frac{-3m(16d^4+32d^3+32d^2+16d)(2^{\lambda_2} \cdot 7^{\lambda_7})}{2^7 \cdot 7} \\
&= \frac{-3m \cdot 16d(d^3+2d^2+2d+1)(2^{\lambda_2} \cdot 7^{\lambda_7})}{2^7 \cdot 7} \\
&= \frac{-3m \cdot 16d(d+1)(d^2+d+1)(2^{\lambda_2} \cdot 7^{\lambda_7})}{2^7 \cdot 7}
\end{aligned}$$

Notice that since we can factor out d, $(d+1)$, and (d^2+d+1) we can infer that any prime can divide into Φ. Knowing that this is the case, by the proof above we have shown that our theory about this system generating all the primes is false. Using a sequence of the largest prime divisors of Φ so that the sequence is monotonic increasing, we can find a known series of primes where the primes are generated by the characteristic, where $p = n(n+1)(n^2+n+1)$ given p is prime and where $n \in \mathbb{Z}$. In conclusion we have shown that this system does not generate the primes, but does produce certain sequences of primes with special characteristics, which hopefully with further research we will discover another sequence with a another special property.

7. Bibliography

References

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[3] Matthias Kreck and Stephan Stolz, "Nonconnected Moduli Spaces of Positive Sectional Curvature Metrics", *Journal of the AMS*, Vol.6, No.4 (Oct., 1993), 825-850.