

# Understanding Animal Group Movement

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## **Abstract**

This paper will explore a few aspects of the field of study that seeks to understand how groups of animals are able to move in a coordinated manner. We will see that efforts in this field are supported by the work of a variety of experts, including mathematicians, computer and related technology specialists, and of course, biologists. Emphasis will be placed on the mathematical modeling of these phenomena and a simple model will be developed.

## **1 Animal Aggregation Study**

The area of study which seeks to understand the details of animal aggregation forms and functions is aided by the achievements of many fields. Among them are those in the following list:

- Animal behavior
- Ecology
- Evolution
- Crystallography
- Geology

- Photogrammetry
- Mathematics

To gain better insight, we ask specific questions of animal grouping phenomena. One set of questions looks at individual actions. In posing these inquiries, we consider the group to be made up of many similar small parts obeying particular grouping rules.

- What are the costs and benefits of group membership?
- Are they positionally dependent?
- What information can, and do, individuals use?
- Do individuals have a sense of the whole?
- Is there an optimal group size?

Another view sees the group as a singular entity; a sort of super-organism. Considerations are centered on describing properties of the group.

- Why are there discrete boundaries?
- What is the appropriate scale for assessing pattern?
- Why should pattern exist in three-dimensional aggregations?
- Is observed three-dimensional structure no more than would result from optimal packing?

Lastly, we want to know how the actions of individuals contribute to the qualities of the group.

- What are the assembly rules?
- Which properties of the group are epiphenomena and which are functional properties that have been selected for?
- Can models which predict epiphenomena be used to make predictions about individual behavior?

Following these prompts, each field involved is able to make valuable contributions to studying animal group behavior. Technological contributions allow us to gather masses of data on large numbers of individuals at once, store the data, and reproduce it in computer simulations or in tables. Limitations in technological ability are overcome by the building of mathematical models to mimic group behavior. Certainly, many laws of physics are used to make realistic models. Such models are made more accurate by a biological understanding of animal behaviors. It is in this way that many areas of study come together to form their own kind of cohesive aggregation.

## 2 Building a Model

Let us now consider the means by which a mathematical model may be constructed. We will model the alignment of fish schools in one dimension. Alignment involves the adjustment of headings of individuals in a school, allowing them to continue to travel with neighbors in a school. In one dimension, we have only two headings, or directions. We label them, arbitrarily, positive and negative. Thus, all fish facing the positive direction are positively aligned.

We call the number of positively aligned fish the positive density,  $u^+$ , and the number of negatively aligned fish is the negative density,  $u^-$ . The total density, or simply, the number of fish in the school, is represented by  $u$ .

As time passes, many fish in a school will change their alignment, based on the relative size of positive density to negative density. So, we include in our model functions  $f^\pm(u^+, u^-)$ , which are the rates of change of the two direction-densities.

We will speak of density states, or states. These are represented first as an ordered pair,  $(u^+, u^-)$ , and later as a single value,  $p = \frac{u^+}{u}$ . They each describe the state of alignment of the school, the ordered pair giving the positive and negative densities, respectively, and the value  $p$  giving the ratio of positive density to total density.

## 2.1 Assumptions

In order to keep our model simple, we make a few assumptions.

- Individuals are identical.
- Individuals follow the same rules for alignment, based on relative sizes of the two direction-densities.
- Individuals move at one constant speed.

## 2.2 The Model

To build our model, we begin with reaction-transport equations.

$$\begin{cases} u_t^+ + \gamma u_x^+ &= f^+(u^+, u^-) - \frac{\mu_*}{2}(u^+ - u^-) \\ u_t^- - \gamma u_x^- &= f^-(u^+, u^-) - \frac{\mu_*}{2}(u^- - u^+) \end{cases}$$

Here, these equations are taken as they are with no development. They describe interaction processes which depend on direction of movement.

Our interest at present is to derive a candidate for our function,  $f(u^+, u^-)$ . This function must meet three specific criteria.

1. Total mass is preserved.

A fish may choose to change direction while remaining with the school. Then, the total group size will not be affected. A decrease in some direction-density yields an equal total increase across all other direction-densities. In one dimension, this means simply that a change in positive density corresponds to an equal and opposite change in negative density.

$$f^+ = -f^- = f$$

and we have simplified our notation as above.

2. Left/right symmetry is satisfied.

If we were to put a school of fish in a tank and build simultaneously a model from observations of that school from the front of the tank and a model from the back of the tank, our final set of equations would be equal and opposite. Essentially, the only difference is that we have changed perspective so that pairs in the domain of our function are in reverse order. This follows naturally from total mass preservation since we are trading positive density for negative density.

$$f(a, b) = -f(b, a)$$

3. Stationary states must not be affected by the alignment process.

$$(u^+, 0), (0, u^-), \left(\frac{u}{2}, \frac{u}{2}\right)$$

are our stationary states.

Considering a fixed school size, there should be no change in the points at which the alignment process ceases. When the fish are positively aligned, that is, when the positive density equals the total density, there will no longer be alignment changes (excepting random turning represented by  $\mu_*$  in the reaction-transport equations). Similarly, total negative alignment yields a cessation in the alignment process. If the fish are tending away from the majority because of a relatively large value of  $\mu_*$ , we get a stationary state at the point where each of our direction-densities is equal to half the total density. In short, values for each of our stationary states are dependent solely on the total density. If this is fixed, so then are the stationary states.

There must be some provision made for changes of school size. We introduce two new functions,  $f_1$  and  $f_2$ , where  $f = f_1 * f_2$ . All characteristics of our density change which are dependent upon school size may be realized with proper selection of  $f_1$ . We will not consider candidates for  $f_1$  in this paper.

Our interest is in working a viable function  $f_2$  into our reaction-transport equations. This will be a function of  $p$  as mentioned above. It will reflect those qualities of alignment that depend upon relative direction-density sizes.

$$p(1 - p)(p - 1/2)$$

meets all the stated criteria.

Substituting for  $p$ ,

$$= \frac{u^+u^-}{u^3}(u^+ - u^-).$$

Now, we return to our reaction-transport equations.

$$\begin{cases} u_t^+ + \gamma u_x^+ &= f^+(u^+, u^-) - \frac{\mu_*}{2}(u^+ - u^-) \\ u_t^- - \gamma u_x^- &= f^-(u^+, u^-) - \frac{\mu_*}{2}(u^- - u^+) \end{cases}$$

We substitute for  $f^+$ ,  $f^-$ .

$$\begin{cases} &= f_1(u)f_2(p) - \frac{\mu_*}{2}(u^+ - u^-) \\ &= -f_1(u)f_2(p) - \frac{\mu_*}{2}(u^- - u^+) \end{cases}$$

Substituting for  $f_2$ :

$$\begin{cases} &= f_1(u)\frac{u^+u^-}{u^3}(u^+ - u^-) - \frac{\mu_*}{2}(u^+ - u^-) \\ &= f_1(u)\frac{u^+u^-}{u^3}(u^- - u^+) - \frac{\mu_*}{2}(u^- - u^+) \end{cases}$$

Now, we let  $a(u) = \frac{f_1(u)}{u^3}$  and factor.

$$\begin{cases} &= [a(u)u^+u^- - \frac{\mu_*}{2}](u^+ - u^-) \\ &= [a(u)u^+u^- - \frac{\mu_*}{2}](u^- - u^+) \end{cases}$$

Finally, we let  $\mu(u^+, u^-) = a(u)u^+u^- - \frac{\mu_*}{2}$ .

Here we have the alignment rate representing the strength of the tendency for the fish to align to the majority,  $a(u)u^+u^-$  competing with the turning rate,  $\mu_*$ , a measure of the amount of random turning exhibited by individuals in the school. When the turning rate is large as compared with the alignment rate, individuals turn toward the minority. Substituting into our equations thus far:

$$\begin{cases} u_t^+ + \gamma u_x^+ &= \mu(u^+, u^-)(u^+ - u^-) \\ u_t^- - \gamma u_x^- &= \mu(u^+, u^-)(u^- - u^+) \end{cases}$$

### **3 Conclusion**

Now we have a model for alignment based on direction of movement which incorporates considerations of local school direction-densities over time. Further development would introduce boundary conditions, speed constraints, characteristics of school size and, especially, movement of the school.

## References

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