NAÏVE DERIVATIVES

AMANDA HOFFMAN
SAM HOUSTON STATE UNIVERSITY

Abstract. Calculus students often think that derivative rules such as the power rule, product rule, and quotient rule are the definition of the derivative and attempt to use these rules for functions to which they do not apply. In this paper, we explore the types of functions that prove these students correct. In particular, when $y = f(x)^{g(x)}$, we ask what conditions f and g must satisfy so that $\frac{dy}{dx} = a{\cdot}g(x)f(x)^{g(x)-1}$. Also, if $y = f(x)g(x)$, we ask what conditions f and g must satisfy so that $\frac{dy}{dx} = a{\cdot}f'(x)g'(x)$. A similar question will be asked for the quotient rule. Specific examples will be given for all cases.

1. Introduction

Many beginning calculus students have misconceived ideas about the method of taking the derivative of certain types of functions. Before learning the Product Rule for derivatives, a student might believe that the derivative of a product of functions is the product of the derivatives of these functions. For example, $(x \ln x)' = \frac{1}{x}$ is an incorrect statement that a student might believe to be true.

Another common calculus error concerns the Power Rule for derivatives. When calculus students are introduced to this rule, they often attempt to use it for functions to which it does not apply. An example of this common calculus error is the claim that $(e^x)' = xe^{x-1}$.

In addition to these problems, calculus students may have misconceived ideas about the derivatives of functions such as $y = \frac{f(x)}{g(x)}$ before they become familiar with the Quotient Rule for derivatives. A common mistake is to claim that the derivative of a quotient of functions is simply the quotient of the derivatives of these functions. For example, a student might incorrectly state that $\left[\frac{x}{(x-3)^5}\right]' = \frac{1}{5(x-3)^4}$.

Although these misconceptions about the derivatives of these certain types of functions are not true in general, there are functions for which these students are correct. In this paper, we will specify conditions on

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f and g so that this "naïve " derivative is equal to the actual derivative. The techniques used for determining these conditions for a naïve product rule and those used in the case of a naïve power rule are very similar and use logarithmic differentiation and elementary differential equations. However, the techniques used to determine conditions for a naïve quotient rule are very different.

2. A Naïve Product Rule

Recall the Product Rule for derivatives from calculus.

Theorem 1. *If f and g are differentiable at x, then fg is also differentiable, and $[fg]' = f'g + fg'$.*

For our naïve product rule, we will find functions f and g so that $y = f(x)g(x)$ has derivative $y' = a{\cdot}f'(x)g'(x)$, where $a \in \mathbb{R}$. Note that we must place the following restrictions on f, g, and a in order to go through this derivation. For the remainder of this section, let f and g be functions such that $g(x) > 0$ and $a{\cdot}f'(x) - f(x) \neq 0$.

Theorem 2. *Let $a, C \in \mathbb{R}$. Suppose that f and g are differentiable real-valued functions. If $y = f(x)g(x)$, then $y' = a{\cdot}f'(x)g'(x)$ if and only if $g(x) = \exp\left(\int \frac{f'(x)}{a\cdot f'(x)-f(x)}dx - C\right)$.*

Proof. Let $y = f(x)g(x)$ and $y' = a{\cdot}f'(x)g'(x)$. From the Product Rule, we have that $y' = f'(x)g(x) + f(x)g'(x)$. Then

$$a{\cdot}f'(x)g'(x) = f'(x)g(x) + f(x)g'(x).$$

It follows that

$$\frac{g'(x)}{g(x)} = \frac{f'(x)}{a{\cdot}f'(x) - f(x)}.$$

We now integrate each side of this equation in order to find $g(x)$ in terms of $f(x)$ and $f'(x)$. In particular,

$$\int \frac{g'(x)}{g(x)}dx = \int \frac{f'(x)}{a{\cdot}f'(x) - f(x)}dx$$

$$\Rightarrow \quad \ln g(x) + C = \int \frac{f'(x)}{a{\cdot}f'(x) - f(x)}dx.$$

Solving for $g(x)$, we find

$$g(x) = \exp\left(\int \frac{f'(x)}{a{\cdot}f'(x) - f(x)}dx - C\right).$$

Thus if $y = f(x)g(x)$ and $y' = a{\cdot}f'(x)g'(x)$, then $g(x) = \exp\left(\int \frac{f'(x)}{a\cdot f'(x)-f(x)}dx - C\right)$.

Now let $y = f(x)g(x)$ and $g(x) = \exp\left(\int \frac{f'(x)}{a\cdot f'(x)-f(x)}dx - C\right)$. Taking the natural logarithm of both sides of the $g(x)$ equation, we find that

$$\ln g(x) = \int \frac{f'(x)}{a\cdot f'(x) - f(x)}dx - C.$$

Differentiating both sides of this equation gives us

$$\frac{g'(x)}{g(x)} = \frac{f'(x)}{a\cdot f'(x) - f(x)}.$$

We then find through algebraic manipulation that

$$a\cdot f'(x)g'(x) = f'(x)g(x) + f(x)g'(x).$$

Since the righthand side of this equation is the actual derivative of $y = f(x)g(x)$, $y' = a\cdot f'(x)g'(x)$. Thus if $y = f(x)g(x)$ and $g(x) = \exp\left(\int \frac{f'(x)}{a\cdot f'(x)-f(x)}dx - C\right)$, then $y' = a\cdot f'(x)g'(x)$.

Therefore if $y = f(x)g(x)$, then $y' = a\cdot f'(x)g'(x)$ if and only if $g(x) = \exp\left(\int \frac{f'(x)}{a\cdot f'(x)-f(x)}dx - C\right)$. □

The following corollaries and examples give us specific pairs of functions f and g which satisfy our naïve product rule. In these corollaries and examples, we will mostly be using the reverse direction of the if and only if statement in this theorem. Notice that the difficulty in finding specific pairs of functions is in calculating the integral in $g(x)$.

Corollary 3. *Let $b, C, D \in \mathbb{R}$. Suppose that $f(x) = b^x$ and $g(x) = \exp\left(\frac{\ln b}{a(\ln b)-1}x + D - C\right)$, where $a \in \mathbb{R}$ but $a \neq \frac{1}{\ln b}$. If $y = f(x)g(x) = [b^x]\left[\exp\left(\frac{\ln b}{a(\ln b)-1}x + D - C\right)\right]$, then*

$$y' = a\cdot f'(x)g'(x) = a\cdot [(\ln b)b^x]\left[\left(\frac{\ln b}{a(\ln b) - 1}\right)\exp\left(\frac{\ln b}{a(\ln b)-1}x + D - C\right)\right].$$

Proof. Note that, with $f(x)$ and $g(x)$ as in the above corollary, we have from the main theorem that

$$\begin{aligned}
&\exp\left(\int \frac{f'(x)}{a\cdot f'(x) - f(x)}dx - C\right)\\
&= \exp\left(\int \frac{(\ln b)b^x}{a(\ln b)b^x - b^x}dx - C\right)\\
&= \exp\left(\int \frac{\ln b}{a(\ln b) - 1}dx - C\right)\\
&= \exp\left(\frac{\ln b}{a(\ln b) - 1}x + D - C\right)\\
&= g(x).
\end{aligned}$$

Thus this corollary follows from the previous theorem. ☐

Example 1. *Suppose* $f(x) = 4^x$ *and we let* $a = 3$ *and* $D - C = 0$*, and therefore we let* $g(x) = \exp\left(\frac{\ln 4}{3(\ln 4)-1}x\right)$ *as in the above corollary. Then* $y = 4^x\left[\exp\left(\frac{\ln 4}{3(\ln 4)-1}x\right)\right]$ *has derivative* $y' = 3\left[(\ln 4)4^x\right]\left[\exp\left(\frac{\ln 4}{3(\ln 4)-1}x\right)\left(\frac{\ln 4}{3(\ln 4)-1}\right)\right]$.

Corollary 4. *Let* $a, b, m, n, C, D \in \mathbb{R}$ *such that* $m \neq 0$*. Suppose that* $f(x) = (mx + b)^n$ *where* $mx + b > 0$ *and* $g(x) = \frac{\exp(D-C)}{|anm-(mx+b)|^n}$*, where* $x \neq \frac{anm-b}{m}$*. If*

$$y = f(x)g(x) = \left[(mx + b)^n\right]\left[\frac{\exp(D - C)}{|anm - (mx + b)|^n}\right],$$

then

$$y' = a\cdot\left[nm(mx + b)^{n-1}\right]\left[g'(x)\right].$$

Proof. Note that, with $f(x)$ and $g(x)$ as in the above corollary, we have from the main theorem that

$$\begin{aligned}
&\exp\left(\int \frac{f'(x)}{a\cdot f'(x) - f(x)}dx - C\right)\\
&= \exp\left(\int \frac{nm(mx + b)^{n-1}}{anm(mx + b)^{n-1} - (mx + b)^n}dx - C\right)\\
&= \frac{\exp(D - C)}{|anm - (mx + b)|^n}\\
&= g(x).
\end{aligned}$$

Thus this corollary follows from the previous theorem. ☐

Example 2. *In particular, if* $f(x) = (3x+5)^5$ *and we let* $a = \frac{1}{15}$ *and* $D - C = 0$*, then* $g(x) = \frac{1}{(-1+(3x+5))^5}$*. Thus if* $y = (3x+5)^5 \frac{1}{(-1+(3x+5))^5}$*, then* $y' = \frac{1}{15}\left[15(3x+5)^4\right]\frac{-15}{(-1+(3x+5))^6}$*.*

3. A Naïve Power Rule

Recall the Power Rule for derivatives from calculus.

Theorem 5. *If* $n \in \mathbb{R}$*, then* $[x^n]' = nx^{n-1}$*.*

For our naïve power rule, we will find functions f and g so that $y = f(x)^{g(x)}$ has derivative $y' = a{\cdot}g(x)f(x)^{g(x)-1}$, where $a \in \mathbb{R}$. Note that we must place the following restrictions on f and g in order to go through this derivation. For the remainder of this section, let f and g be functions such that $f(x) > 0$, $g(x) > 0$, and $f(x) \neq 1$.

Theorem 6. *Let* $a, C \in \mathbb{R}$*. Suppose that* f *and* g *are differentiable real-valued functions such that* $f(x) > 0$*,* $g(x) > 0$*, and* $f(x) \neq 1$*. If* $y = f(x)^{g(x)}$*, then* $y' = a{\cdot}g(x)f(x)^{g(x)-1}$ *if and only if* $g(x) = \exp\left(\int \frac{a-f'(x)}{f(x)\ln f(x)}dx - C\right)$*.*

Proof. Let $y = f(x)^{g(x)}$ and $y' = a{\cdot}g(x)f(x)^{g(x)-1}$.

$$y = f(x)^{g(x)} \Rightarrow \ln y = g(x)\ln f(x).$$

Then by implicit differentiation,

$$\frac{1}{y}y' = \frac{g(x)}{f(x)}f'(x) + g'(x)\ln f(x).$$

Substituting $y = f(x)^{g(x)}$ back into our derivative, we find

$$y' = f(x)^{g(x)}\left[\frac{g(x)}{f(x)}f'(x) + g'(x)\ln f(x)\right].$$

Then

$$y' = a{\cdot}g(x)f(x)^{g(x)-1}$$

$$\Rightarrow f(x)^{g(x)}\left[\frac{g(x)}{f(x)}f'(x) + g'(x)\ln f(x)\right] = a{\cdot}g(x)f(x)^{g(x)-1}.$$

Dividing both sides of this equation by $f(x)^{g(x)}$, we find

$$\frac{g(x)f'(x)}{f(x)} + g'(x)\ln f(x) = \frac{a{\cdot}g(x)}{f(x)}.$$

Manipulating this equation algebraically, we are able to find $\frac{g'(x)}{g(x)}$ in terms of $f(x)$ and $f'(x)$:

$$\frac{g'(x)}{g(x)} = \frac{a - f'(x)}{f(x)\ln f(x)}.$$

We can then integrate each side of this equation in order to solve this equation for $g(x)$ in terms of $f(x)$ and $f'(x)$. In particular,

$$\int \frac{g'(x)}{g(x)}dx = \int \frac{a - f'(x)}{f(x)\ln f(x)}dx$$

$$\Rightarrow \ln g(x) + C = \int \frac{a - f'(x)}{f(x)\ln f(x)}dx.$$

Thus

$$g(x) = \exp\left(\int \frac{a - f'(x)}{f(x)\ln f(x)}dx - C\right).$$

Thus if $y = f(x)^{g(x)}$ and $y' = a{\cdot}g(x)f(x)^{g(x)-1}$, then $g(x) = \exp\left(\int \frac{a-f'(x)}{f(x)\ln f(x)}dx - C\right)$.

Now let $y = f(x)^{g(x)}$ and $g(x) = \exp\left(\int \frac{a-f'(x)}{f(x)\ln f(x)}dx - C\right)$. By taking the natural logarithm of both sides, we find that

$$\ln g(x) = \int \frac{a - f'(x)}{f(x)\ln f(x)}dx - C.$$

Differentiating both sides of this equation,

$$\frac{g'(x)}{g(x)} = \frac{a - f'(x)}{f(x)\ln f(x)}.$$

Through algebraic manipulation and multiplying the equation by $f(x)^{g(x)}$, we find that

$$\Rightarrow f(x)^{g(x)}\left[\frac{g(x)}{f(x)}f'(x) + g'(x)\ln f(x)\right] = a{\cdot}g(x)f(x)^{g(x)-1}.$$

Since the lefthand side of this equation is the actual derivative of $f(x)^{g(x)}$,

$$y' = a{\cdot}g(x)f(x)^{g(x)-1}.$$

Thus if $y = f(x)^{g(x)}$ and $g(x) = \exp\left(\int \frac{a-f'(x)}{f(x)\ln f(x)}dx - C\right)$, then $y' = a{\cdot}g(x)f(x)^{g(x)-1}$.

Therefore if $y = f(x)^{g(x)}$, then $y' = a{\cdot}g(x)f(x)^{g(x)-1}$ if and only if $g(x) = \exp\left(\int \frac{a-f'(x)}{f(x)\ln f(x)}dx - C\right)$. $\square$

The following corollaries and examples give us specific pairs of functions f and g which satisfy our naïve power rule, using this theorem.

Corollary 7. *Let* $a, b, m, C, D \in \mathbb{R}$*, such that* $m \neq 0$*. Suppose that* $f(x) = mx + b$ *and* $g(x) = \exp(D - C)\ln(mx + b)^{\frac{a-m}{m}}$*. If* $y = (mx + b)^{\exp(D-C)\ln(mx+b)^{\frac{a-m}{m}}}$*, then* $y' = a \cdot \left[\exp(D - C)\ln(mx + b)^{\frac{a-m}{m}}\right](mx + b)^{\exp(D-C)\ln(mx+b)^{\frac{a-m}{m}}-1}$.

Proof. Notice that, with $f(x)$ and $g(x)$ as in the above corollary, we have from the main theorem in this section that

$$\begin{aligned}
&\exp\left(\int \frac{a - f'(x)}{f(x)\ln f(x)}dx - C\right) \\
&= \exp\left(\int \frac{a - m}{(mx + b)\ln(mx + b)}dx - C\right) \\
&= \exp\left(\frac{a - m}{m}\ln(\ln(mx + b)) + D - C\right) \\
&= \exp(D - C)\ln(mx + b)^{\frac{a-m}{m}} \\
&= g(x).
\end{aligned}$$

Thus this corollary follows from the previous theorem. □

Example 3. *Suppose* $f(x) = (mx + b)$ *and we let* $D - C = 0$ *and* $a = 2m$*, and therefore we let* $g(x) = \ln(mx + b)$ *as in the above corollary. Then* $y = (mx + b)^{\ln(mx+b)}$ *has derivative* $y' = 2m\ln(mx + b)(mx + b)^{\ln(mx+b)-1}$.

Corollary 8. *Let* $a, b, C, D \in \mathbb{R}$*. Suppose that* $f(x) = b$ *and* $g(x) = \exp\left(\frac{a}{b\ln b}x + D - C\right)$*. If* $y = f(x)^{g(x)} = (b)^{\exp\left(\frac{a}{b\ln b}x+D-C\right)}$*, then* $y' = a \cdot g(x)f(x)^{g(x)-1}$.

Proof. Note that, with $f(x)$ and $g(x)$ as in the above corollary, we have from the main theorem in this section that

$$\begin{aligned}
&\exp\left(\int \frac{a - f'(x)}{f(x)\ln f(x)}dx - C\right) \\
&= \exp\left(\int \frac{a}{b\ln b}dx - C\right) \\
&= \exp\left(\frac{a}{b\ln b}x + D - C\right) \\
&= g(x).
\end{aligned}$$

Thus this corollary follows from the previouis theorem. ☐

Example 4. *Suppose* $f(x) = 2$ *and we let* $D - C = 0$ *and* $a = 2\ln 2$*, and therefore we let* $g(x) = e^x$ *as in the above corollary. Then* $y = 2^{e^x}$ *has derivative* $y' = (2\ln 2)\,(e^x)\,2^{e^x - 1}$.

Note that it is difficult to find $f(x)$ for which we can evaluate $\int \frac{a - f'(x)}{f(x)\ln f(x)}dx$, which shows up in the $g(x)$ of our naïve power rule theorem.

4. A NAÏVE QUOTIENT RULE: AN OPEN PROBLEM

Recall the Quotient Rule from calculus.

Theorem 9. *If* f *and* g *are differentiable at* x *and* $g(x) \neq 0$*, then the quotient* $\frac{f}{g}$ *is differentiable at* x*, and* $\left[\frac{f}{g}\right]' = \frac{f'g - fg'}{g^2}$.

For a naïve quotient rule, we want to find functions f and g so that $y = \frac{f(x)}{g(x)}$ has derivative $y' = a \cdot \frac{f'(x)}{g'(x)}$, where $a \in \mathbb{R}$. Since we want $\left[\frac{f(x)}{g(x)}\right]' = a \cdot \frac{f'(x)}{g'(x)}$,

$$a \cdot \frac{f'(x)}{g'(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Through algebraic manipulation, we find

$$f(x)\left(\frac{g'(x)}{g(x)}\right)^2 - f'(x)\left(\frac{g'(x)}{g(x)}\right) + af'(x) = 0.$$

Solving this quadratic equation in $\frac{g'(x)}{g(x)}$,

$$\frac{g'(x)}{g(x)} = \frac{f'(x) \pm \sqrt{(f'(x))^2 - 4af'(x)f(x)}}{2f(x)}.$$

If we can integrate both sides of this equation, then we can solve for $g(x)$ and find that

$$g(x) = exp\left(\int \frac{f'(x) \pm \sqrt{(f'(x))^2 - 4af'(x)f(x)}}{2f(x)}dx - C\right).$$

This integral brings up several interesting questions. What does it mean for this integral to have a plus or minus sign in it and does it matter? Are there any $f(x)$ for which we can integrate this function? We will continue to research these questions.

5. Conclusions

In our research on naïve derivatives, we have been able to find pairs of functions f and g that satisfy a naïve product rule and pairs of functions that satisfy a naïve power rule. We are still working on a naïve quotient rule; in particular, finding examples satisfying the conditions we have found. In the future, we will examine other derivative rules in order to develop additional naïve derivative rules.