Fourier Transforms and 4-D Tensor-Based Wave Equations

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May 6, 2008

Abstract

Using the well known Fourier Transform, we offer a method of solution for 4-dimensional tensor-based wave equations with time-dependent sources. This method is restricted to waveforms and sources in cartesian coordinates and that tend toward 0 at the infinite limits of its three spatial coordinates.

1 Introduction

In modern physics, the use of tensors is commonplace. Most theories can be written in a tensor-based form. This, in fact, yields a geometric sense to most physical problems. Another familiar mathematical model that arises in nature is the wave equation. This model appears in both Maxwell's and Einstein's theories, however we find that the sources and the waveforms are tensors. The problem is, how do we solve the wave equation with a tensor source and waveform?

In most first courses in differential equations, Laplace transforms are introduced. It is found the these transforms can simplify the equation; taking a differential equation to an algebraic equation. However, because we are going to be taking solutions over all of space, the Laplace transform will not do. Turning to a very similar transform, the Fourier transform, we find that in certain situations it may be used for solving PDEs. [1] Many properties of these transforms make them ideal for working with PDEs, specifically their derivative properties. [2] Because of this, our focus will be on applying the Fourier transform to the tensor based wave equation in four dimensions.

2 Tensors and their Properties

An $n-$th rank tensor in $m-$dimensional space is a mathematical object that has n indices and m^n components and obeys the following transformation rule:

$$S^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots} = \frac{\partial z^{\mu_1}}{\partial x^{\alpha_1}}\frac{\partial z^{\mu_2}}{\partial x^{\alpha_2}}\cdots\frac{\partial x^{\beta_1}}{\partial x^{\nu_1}}\frac{\partial x^{\beta_2}}{\partial z^{\nu_2}}\cdots S^{\alpha_1\alpha_2\cdots}_{\beta_1\beta_2\cdots}$$

It can also be shown that zeroth-rank, first-rank and second-rank tensors are, in fact, scalars, vectors, and matrices. Now, for the sake of notation we will limit ourselves to rank$-(0,2)$ tensors. However, this is not a requirement. We will be performing operations on the elements of tensor, not on the tensors themselves. This allows for us to extend this method to tensors of arbitrary rank.

Now, let us consider our wave equation:

$$S_{\mu\nu,\alpha}{}^{\alpha} = T_{\mu\nu}\,.$$

Remember now that we are dealing with 4−space so each index takes on values from 0 to 4. Limiting ourselves to cartesian coordinates and expanding yeilds:

$$-\frac{1}{c^2}\frac{\partial^2 S_{\mu\nu}}{\partial t^2}+\frac{\partial^2 S_{\mu\nu}}{\partial x^2}+\frac{\partial^2 S_{\mu\nu}}{\partial y^2}+\frac{\partial^2 S_{\mu\nu}}{\partial z^2}=T_{\mu\nu}\,. \tag{1}$$

Finally we will restrict $S_{\mu\nu}$ to functions whose infinite limits are 0. That is:

$$\lim_{x\to\infty} S_{\mu\nu} = \lim_{y\to\infty} S_{\mu\nu} = \lim_{z\to\infty} S_{\mu\nu} = 0$$

The reasoning for this follows from the next section.

3 Useful Properties of the Fourier Transform

It is easily seen that the Fourier transform strongly resembles the Laplace transform. However, you see that the Fourier transform has complex exponents, and has limits of integration of $\pm\infty$. Because our waves are in 4-space, we want to take the Fourier transform of the wave equation in 3-dimensions. Here,

$$\mathbf{F}(t,\alpha,\beta,\gamma)=\mathcal{F}\left\{f(t,x,y,z);(t,\alpha,\beta,\gamma)\right\}=\int\int\int_{-\infty}^{\infty} f(t,x,y,z)e^{2\pi i(\alpha x+\beta y+\gamma z)}dx\ dy\ dy \tag{2}$$

and

$$f(t,x,y,z)=\mathcal{F}^{-1}\left\{\mathbf{F}(t,\alpha,\beta,\gamma);(t,x,y,z)\right\}=\int\int\int_{-\infty}^{\infty} \mathbf{F}(t,\alpha,\beta,\gamma)e^{-2\pi i(\alpha x+\beta y+\gamma z)}d\alpha\ d\beta\ d\gamma \tag{3}$$

are the Fourier transform and inverse Fourier transform, respectively. We will denote a transformed function with boldface type.

Now, the Fourier transform and inverse Fourier transform have the same following properties.

Linearity will give us the ability to break our equations into smaller pieces. Simply the idea of divide and conquer:

$$\mathcal{F}\left\{a\cdot f(x)+b\cdot g(x);\ \omega\right\}=a\cdot\mathbf{F}(\omega)+b\cdot\mathbf{G}(\omega). \tag{4}$$

Because our waveform will be approaching zero in the infinite limits of its spatial coordinates, the transform across our spatial coordinates of the derivatives of the waveform with respect to those coordinates yields:

$$\mathcal{F}\left\{\frac{\partial^n}{\partial x^n}f(t,x);\ (t,\omega)\right\}=(2\pi\imath\omega)^n\mathbf{F}(t,\omega). \tag{5}$$

This property will allow us to use the Fourier transform to convert our PDE to and ODE.

Finally, the Fourier transform across the spatial components of the derivative of our waveform with respect to time will give:

$$\mathcal{F}\left\{\frac{\partial^2}{\partial t^2}f(t,x);\ (t,\omega)\right\}=\frac{\partial^2}{\partial t^2}\mathbf{F}(t,\omega). \tag{6}$$

This simply means that if we apply to transform to our equation across space alone, we will retain our time derivative.

Applying the Fourier Transform to the Wave Equation

Now, we are ready to begin our solution. Let us start by stating the wave equation in an extended and general form:

$$-\frac{1}{v^2}\frac{\partial^2 S_{\mu\nu}}{\partial t^2}+\frac{\partial^2 S_{\mu\nu}}{\partial x^2}+\frac{\partial^2 S_{\mu\nu}}{\partial y^2}+\frac{\partial^2 S_{\mu\nu}}{\partial z^2}=T_{\mu\nu} \tag{7}$$

Applying the Fourier transform to both sides and utilizing Properties 4, 5, and 6 gives us:

$$\frac{d^2}{dt^2}\mathbf{S}_{\mu\nu}+4\pi^2v^2\left(\alpha^2+\beta^2+\gamma^2\right)\mathbf{S}_{\mu\nu}=-v^2\mathbf{T}_{\mu\nu} \tag{8}$$

Now, we will use the method of variation of parameters. To do so, solve the homogeneous equation:

$$\frac{d^2}{dt^2}\mathbf{S}^*_{\mu\nu}-4\pi^2v^2\left(\alpha^2+\beta^2+\gamma^2\right)\mathbf{S}^*_{\mu\nu}=0$$

It can be shown that the solution to this ODE is of the form:

$$\mathbf{S^*}_{\mu\nu}=c_1e^{2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}}+c_2e^{-2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

Now, any solution to Equation 8 will be of the form:

$$\mathbf{S}_{\mu\nu}=v_1\left(t,\alpha,\beta,\gamma\right)e^{2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}}+v_2\left(t,\alpha,\beta,\gamma\right)e^{-2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}} \tag{9}$$

Applying variation of parameters yields:

$$v_1=\int\frac{-v^2\mathbf{T}_{\mu\nu}e^{-2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}}}{\mathbf{W}[y_1,y_2]}\,dt \tag{10}$$

$$v_2=\int\frac{v^2\mathbf{T}_{\mu\nu}e^{2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}}}{\mathbf{W}[y_1,y_2]}\,dt \tag{11}$$

where

$$\mathbf{W}[y_1,y_2]=\begin{vmatrix} e^{2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}} & e^{-2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}} \\ \frac{d}{dt}\left\{e^{2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}}\right\} & \frac{d}{dt}\left\{e^{-2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}}\right\} \end{vmatrix}$$

or, more simply, we have that

$$\mathbf{W}[y_1,y_2]=4\pi iv\sqrt{\alpha^2+\beta^2+\gamma^2}.$$

Upon substituting for v_1 and v_2 in Equation 9 gives us:

$$\mathbf{S}_{\mu\nu}=e^{2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}}\int\frac{-v^2\mathbf{T}_{\mu\nu}e^{-2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}}}{4\pi iv\sqrt{\alpha^2+\beta^2+\gamma^2}}\,dt+e^{-2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}}\int\frac{v^2\mathbf{T}_{\mu\nu}e^{2\pi ivt\sqrt{\alpha^2+\beta^2+\gamma^2}}}{4\pi iv\sqrt{\alpha^2+\beta^2+\gamma^2}}\,dt. \tag{12}$$

Finally, applying an inverse Fourier transform to both sides leaves us with:

$$
\begin{aligned}
S_{\mu\nu} = & \int\int\int_{-\infty}^{\infty} e^{2\pi i v t\sqrt{\alpha^2+\beta^2+\gamma^2}} e^{-2\pi i(\alpha x+\beta y+\gamma z)} \int \frac{-v^2 \mathbf{T}_{\mu\nu} e^{-2\pi i v t\sqrt{\alpha^2+\beta^2+\gamma^2}}}{4\pi i v\sqrt{\alpha^2+\beta^2+\gamma^2}}\, dt\, d\alpha\, d\beta\, d\gamma \\
& + \int\int\int_{-\infty}^{\infty} e^{-2\pi i v t\sqrt{\alpha^2+\beta^2+\gamma^2}} e^{-2\pi i(\alpha x+\beta y+\gamma z)} \int \frac{v^2 \mathbf{T}_{\mu\nu} e^{2\pi i v t\sqrt{\alpha^2+\beta^2+\gamma^2}}}{4\pi i v\sqrt{\alpha^2+\beta^2+\gamma^2}}\, dt\, d\alpha\, d\beta\, d\gamma
\end{aligned} \tag{13}
$$

4 Conclusion

It is clear that any real source function can be used in this method, yet we see from from Equation 13 that not just any source function will produce valuable results. Specifically, if the Fourier transform of our source function makes the integrals in Equation 12 impossible, then this method fails. However, we also find that, in some special cases the solution reduces. An example of such a case is any constant source function. In the future it is hoped that this method can be extended to non-cartesian coordinates.

References

[1] A. P. S. Selvadurai. *Partial differential equations in mechanics. 1.* Springer-Verlag, Berlin, 2000. Fundamentals, Laplace's equation, diffusion equation, wave equation.

[2] M. Spiegel. *Fourier Analysis with Applications to Boundary Value Problems".* McGraw-Hill, 1974.