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%\renewcommand{\baselinestretch}{1.4}
\usepackage{amsmath}
\usepackage{amssymb}
\usepackage{amscd}
\usepackage{pdfsync}

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\def\R{\mathbb R}
\def\im{{\rm im \,}}
\def\ra{\rightarrow}
\def\lra{\longrightarrow}
\def\la{\leftarrow}
\def\<{\langle}
\def\>{\rangle}
\def\lcm{\mbox{lcm}}
\def\supp{\rm supp \,}
\newcommand{\tfr}[1]{\mbox{\rm rk}_{\Z}(#1)}
\newcommand{\Ann}[1]{\mbox{\rm Ann}\,_{\Z #1}}\,
\newcommand{\Res}[2]{\mbox{\rm Res}\,^{\Z #1}_{\Z #2}}\,
\newcommand{\Cent}[1]{\mbox{\rm Cent}\,_{\Z #1}}\,
\newcommand{\Sig}[2]{\Sigma({#1},{#2})}
\newcommand{\Sigc}[2]{\Sigma({#1},{#2})^c}
\newcommand{\ov}[1]{\overline{#1}}
\def\Proof{\noindent {\bf Proof:}\ }
\def\qed{\hfill $\bullet$}
\renewcommand{\tilde}{\widetilde}
\renewcommand{\bar}{\overline}


\title{\Large On the Homotopy Type of CW-Complexes with Aspherical
Fundamental Group}

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\date{\small January 2005}
\pagestyle{plain}
\newtheorem{theorem}{Theorem}[section]
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\newtheorem{proposition}[theorem]{Proposition}
\newtheorem{lemma}[theorem]{Lemma}
\newtheorem{corollary}[theorem]{Corollary}
\newtheorem{example}[theorem]{Example}
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\begin{document}
\maketitle

\begin{abstract} This paper is concerned with the homotopy classification of finite CW-complexes. A (G,n)-complex is a finite n-dimensional CW-complex with fundamental-group G and
vanishing higher homotopy-groups up to dimension $n-1$. In case G is finite dimensional there is a unique (up to homotopy) (G,n)-complex on the minimal Euler-characteristic level $\chi_{min}(G,n)$. We show that if the finite dimensional group G contains the trefoil group T as a retract then there is more than one homotopy-type on the level $\chi_{min}(G,n)+1$. We also outline a program for constructing different homotopy types of 2-complexes on Euler-characteristic levels higher that $\chi_{min}(G,n)+1$.

\end{abstract}

\medskip

\noindent {\small AMS Subject classification: Primary 57M20, Secondary 57M05.
\newline Keywords: 2-dimensional complex, homotopy-type, stably-free modules}

\section{Introduction}

This paper is concerned with the homotopy classification of CW-complexes. A CW-complex is called {\sl aspherical} if all its higher homotopy groups vanish. A (G,n)-complex is a finite n-dimensional CW-complex with fundamental-group G and vanishing higher homotopy-groups up to dimension $n-1$. Note that a (G,n)-complex is the n-skeleton of an aspherical complex that has finite n-skeleton. Note also that a $(G,2)$-complex is simply a finite 2-complex with fundamental group G. For a given group G we are investigating the question whether there can be homotopically distinct (G,n)-complexes with the same Euler-characteristic.

Suppose that X is a finite n-dimensional CW-complex and denote by c_k the number of k-cells of X. By the {\sl directed Euler-characteristic of X}, $\chi_d(X)$, we mean the alternating sum $\sum_{i=0}^n (-1)^{n-i}c_{i}$. If X is a (G,n)-complex then it is not difficult to see that $\chi_d(X)$ is bounded from below by $\sum_{i=0}^n (-1)^{n-i}\textrm{dim} H_i(G,\mathbb Q)$, a constant that only depends on the homology of G. Thus we can define $\chi_{\min}(G,n)$ to be the minimal directed Euler-characteristic that can occur.

We say a group G is {\sl n-dimensional} if it is the fundamental-group of a finite n-dimensional aspherical complex and there is no such complex of smaller dimension. Every (G,n)-complex of minimal directed Euler-characteristic $\chi(G,n)$ of a n-dimensional group G is aspherical. Hence up to homotopy there is a unique (G,n)-complex of minimal directed Euler-characteristic (Theorem 3.1). We show in this paper that if G is a n-dimensional group that contains the trefoil-group as a retract, then there are homotopically inequivalent (G,n)-complexes with directed Euler-characteristic $\chi_{min}(G,n)+1$ (Theorem 3.5). For the trefoil-group itself this was observed by Dunwoody \cite{Dunwoody}. See also the interesting generalizations obtained by Lustig \cite{Lustig}. We also outline a program for constructing different homotopy types of 2-complexes on Euler-characteristic levels higher that $\chi_{min}(G,n)+1$ (Theorem 3.6 and Section 5). Additional information on the classification of homotopy types and related topics can be found in the excellent book \cite{metzlerbook}.

\section{Presentations of stably-free modules}

Let R be a unitary ring. A R-module P is called stably-free if there are natural numbers m and n so that $P\oplus R^m$ is isomorphic to R^n. Another way to say this is that a stably-free module is the kernel of an epimorphism $\phi: R^n \rightarrow R^m$. By a {\sl splitting of ϕ} we mean a homomorphism $s: R^m \rightarrow R^n$ such that $\phi\circ s$ is the identity.

\begin{lemma} {Let P be the kernel of an epimorphism $\phi: R^n \rightarrow R^m$. Choose a basis $e_1, \ldots ,e_n$ of R^n and a splitting s of ϕ. Then P is generated by the elements $e_i-s\circ \phi(e_i)$, $i=1,\ldots ,n$. Furthermore the inclusion induces an isomorphism $P\rightarrow R^n/s(R^m)$.}
\end{lemma}

\Proof Since every element v of R^n can be uniquely written as $$v=(v-s\circ \phi(v))\oplus s\circ \phi(v)$$ we see that $R^n=P\oplus s(R^n)$. Since the elements e_i, $i=1,\ldots ,n$ generate R^n and $$e_i=(e_i-s\circ \phi (e_i))\oplus s\circ \phi (e_i)$$ we see that the elements $e_i-s\circ \phi (e_i)$, $i=1, \ldots ,n$, generate P and that the inclusion induces an isomorphism $P\rightarrow R^n/s(R^m)$. \qed

\bigskip
Notice that if $m=1$ and $\phi(e_i)=\alpha_i \in R$, $i=1,\ldots ,n$, every choice of elements β_i, $i=1, \ldots ,n$, such that $\sum_{i=1}^n \beta_i\alpha_i=1$ determines a splitting of ϕ. Indeed, simply define $s(1)=\sum_{i=1}^n \beta_ie_i$.

\bigskip
In the remainder of this section we will discuss Dunwoody's exotic presentation for the trefoil group T (see \cite{Dunwoody}). First, T has the well known 1-relator presentation $\langle a, b, \mid a^2=b^3 \rangle$. Let X be the 2-complex associated with it. Let $r=a^2b^{-3}$ and denote by N the normal closure of r in the free group on a,b. Dunwoody considers the presentation $\langle a, b \mid u_1, u_2 \rangle$ where $u_1=rara^{-1}a^2ra^{-2}$ and $u_2=rbrb^{-1}b^2rb^{-2}b^3rb^{-3}$ and shows that the second homotopy module $\pi_2(X_1)$ of the associated 2-complex X_1 can not be generated by a single element and hence is stably-free but not free. Since the presentation $\langle a, b, \mid a^2=b^3, 1 \rangle$ gives rise to a 2-complex X_2 with second homotopy module free of rank one, we see that there are homotopically distinct $(T,2)$-complexes with Euler-characteristic $\chi_{\min}(T,2)+1=1$.

Using the above Lemma 2.1 it is not difficult to exhibit generators and a presentation for the module $\pi_2(X_1)$. Let $\alpha_1=1+a+a^2$ and $\alpha_2=1+b+b^2+b^3$. Consider the cellular chain complex $(C_*(\tilde X_1),\partial)$ of the universal covering of X_1. It gives rise to an exact sequence (see \cite{LyndonSchupp}, Section 3 of Chapter II) $$0\rightarrow \pi_2(X_1)\rightarrow C_2(\tilde X_1) \stackrel{\phi}{\rightarrow} \bar N\rightarrow 0$$ where $\bar N$ is the relation-module for the generators a,b of T. It is free of rank 1 and is generated by $r[N,N]$. The second chain group $C_2(\tilde X)$ has a basis e_1, e_2 consisting of 2-cells that present lifts of the 2-cells in X corresponding to the two relations $u_1=rara^{-1}a^2ra^{-2}$ and $u_2=rbrb^{-1}b^2rb^{-2}b^3rb^{-3}$. Furthermore

$\phi(e_i)=u_i[N,N]=\alpha_ir[N,N]$, $i=1,2$. Lemma 2.1 and the
remark thereafter tell us that every choice of elements $\beta_1,
\beta_2 \in \mathbb Z T$ such that
$\beta_1\alpha_1+\beta_2\alpha_2=1$ gives rise to a splitting of
ϕ and hence to explicit generators
$e_i-\alpha_i(\beta_1e_1+\beta_2e_2)$ and a presentation for
$\pi_2(X_1)=\mathbb Z T^2/\beta_1e_1+\beta_2e_2$.

In the following we will compute a particular choice for β_1
and β_2. Note first that $(a-1)\alpha_1=a^3-1$ and
$(b-1)\alpha_2=b^4-1$. Set $x=a^3$ and $y=b^4$. The elements x
and y generate the group. Indeed, $x^3y^{-3}=a$ and
$x^2y^{-2}=b$. Hence $x-1$ and $y-1$ generate the augmentation
ideal IT. Since $\alpha_1-\alpha_2$ augments to -1 we see that
$x-1$, $y-1$ and $\alpha_1-\alpha_2$ generate $\mathbb Z T$ and
hence α_1 and α_2 generate $\mathbb Z T$. Now
$\alpha_1-\alpha_2+1$ is in the augmentation ideal IT and so we
can write it as a linear combination
$\alpha_1-\alpha_2+1=\gamma_1(x-1)+\gamma_2(y-1)$ for certain
$\gamma_i \in \mathbb Z T$. Solving for 1 we obtain
$1=(\gamma_1(a-1)-1)\alpha_1+(\gamma_2(b-1)+1)\alpha_2$. So we get
a choice for the desired β_i by computing the γ_i and
that can be quickly accomplished using the Fox-calculus (see
\cite{LyndonSchupp}, Section 3 of Chapter II).
\begin{eqnarray*}
\alpha_1-\alpha_2+1 &=& (a-1)-(b+2)(b-1) \\
&=& (x^3y^{-3}-1)-(b+2)(x^2y^{-2}-1) \\
&=& \frac{\partial x^3y^{-3}}{\partial x}(x-1)+\frac{\partial x^3y^{-3}}
{\partial
y}(y-1)\\
& & -(b+2)\left(\frac{\partial x^2y^{-2}}{\partial x}(x-1)+\frac{\partial x^2y^
{-2}}{\partial
y}(y-1)\right) \\
&=& \left(\frac{\partial x^3y^{-3}}{\partial
x}-(b+2)\frac{\partial x^2y^{-2}}{\partial
x}\right)(x-1) \\
& & +\left(\frac{\partial x^3y^{-3}}{\partial
y}-(b+2)\frac{\partial x^2y^{-2}}{\partial
y}\right)(y-1).
\end{eqnarray*}
Let us make the Fox-derivatives explicit, remembering that $a^2=b^3$ in $\mathbb
Z T$:
$$\frac{\partial x^3y^{-3}}{\partial x}=1+x+x^2=1+a^3+a^6,$$
$$\frac{\partial x^3y^{-3}}{\partial y}=-x^3y^{-1}-x^3y^{-2}-x^3y^{-3}=-(a+ab^4
+ab^8).$$
Similarly we get

$$\frac{\partial x^2y^{-2}}{\partial x}=1+a^3,$$
$$\frac{\partial x^2y^{-2}}{\partial y}=-(b+b^5).$$
Thus we have
$$\gamma_1=(1+a^3+a^6)-(b+2)(1+a^3),$$
$$\gamma_2=-(a+ab^4+ab^8)+(b+2)(b+b^5),$$
and hence
$$\beta_1=((1+a^3+a^6)-(b+2)(1+a^3))(a-1)-1,$$
$$\beta_2=(-(a+ab^4+ab^8)+(b+2)(b+b^5))(b-1)+1.$$
We summarize our findings in the following

\begin{theorem} Let X_1 be the 2-complex associated with the presentation $$\langle a, b \mid u_1, u_2 \rangle$$ for the trefoil group T, where
$$u_1=rara^{-1}a^2ra^{-2}, u_2=rbrb^{-1}b^2rb^{-2}b^3rb^{-3},$$
$r=a^2b^{-3}$. Then the
second homotopy-module $\pi_2(X_1)$ can not be generated by a single element and hence is stably-free but not free (Dunwoody \cite{Dunwoody}). It is generated as a submodule of $C_2(\tilde X_1)$ by $e_i-\alpha_i(\beta_1e_1+\beta_2e_2)$, $i=1,2$. Furthermore the inclusion $\pi_2(X_1)\hookrightarrow C_2(\tilde X_1)$ induces an isomorphism $\pi_2(X_1)=\mathbb Z T^2/\beta_1e_1+\beta_2e_2$. Here
$$\alpha_1=1+a+a^2, \ \alpha_2=1+b+b^2+b^3$$
$$\beta_1=((1+a^3+a^6)-(b+2)(1+a^3))(a-1)-1,$$
$$\beta_2=(-(a+ab^4+ab^8)+(b+2)(b+b^5))(b-1)+1.$$
\end{theorem}

\bigskip
We end this section with a question. Let X be the 2-complex modelled on the standard one-relator presentation of T and X_1 be as in Theorem 2.2. Let $X_2=X\vee S^2$. Is $Y_1=X_1\vee X_1$ homotopically equivalent to $Y_2=X_2 \vee X_2$? Note that if $G=T*T$ then $\chi(Y_1)=\chi(Y_2)=\chi_{min}(G,2)+2$. So far no pair of homotopically distinct 2-complexes with the same fundamental group and Euler characteristic more than one above the minimal level is known! The question comes down to proving that $\pi_2(Y_1)=\mathbb Z G^4/(\beta_1e_1+\beta_2e_2, \beta_1e_3+\beta_2e_4)$ is not free of rank two where β_1 and β_2 are as in Theorem 2.2.

\section{General results and examples}

\begin{theorem} Let G be a k-dimensional group. Up to homotopy there exists a unique (G,k)-complex with directed Euler characteristic equal to $\chi_{\min}(G,k)$.

\end{theorem}

\Proof Since we assumed G to be k-dimensional there is a finite aspherical k-dimensional complex X with fundamental group G. Since the homology of X is the homology of the group G we have $\chi_d(X)=\chi_{\min}(G,k)$. Suppose Y is a (G,k)-complex with the same Euler characteristic. We will show that Y is aspherical and hence homotopic to X.

Consider the cellular chain complexes $C_*(\tilde X)$ and $C_*(\tilde Y)$ of the universal coverings. It follows from Schanuel's Lemma (see \cite{Brown}) that $H_k(\tilde Y)\oplus A = B$ where
$$A=C_k(\tilde X)\oplus C_{k-1}(\tilde Y)\oplus C_{k-2}(\tilde X)\oplus ...$$ and
$$B=C_k(\tilde Y)\oplus C_{k-1}(\tilde X)\oplus C_{k-2}(\tilde Y)\oplus ...$$ The fact that $\chi_d(X)=\chi_d(Y)$ implies that the free $\mathbb Z G$-modules A and B have equal rank, so $H_k(\tilde Y)\oplus \mathbb Z G^l=\mathbb Z G^l$ for some $l\ge 0$. Kaplansky's Theorem (see \cite{metzlerbook}, page 328) now implies that $H_k(\tilde Y)=0$. So Y is indeed aspherical. \qed

\begin{theorem} Let G be a k-dimensional group, $k\ge 3$, and assume P is a stably-free non-free projective module over the group ring $\mathbb Z G$ which is the kernel of an epimorphism $\phi: \mathbb Z G^n \rightarrow \mathbb Z G^m$. Then there are (G,k)-complexes X_1 and X_2 with directed Euler-characteristic $\chi_{\min}(G,k)+n-m$ such that $\pi_k(X_1)$ is isomorphic to P and $\pi_2(X_2)$ is free of rank $n-m$. In particular X_1 and X_2 are not homotopically equivalent.
\end{theorem}

\Proof Let X be a finite aspherical complex of dimension k with fundamental group G. Consider the left end of the cellular chain complex $(C_*(\tilde X),\partial)$ of the universal covering
$$0\rightarrow C_k(\tilde X)\stackrel{\partial_k}{\rightarrow} C_{k-1}(\tilde X)\rightarrow ...$$ Let $\bar e_1, \ldots ,\bar e_l$ be the k-cells in X and denote by e_i a fixed lift of $\bar e_i$ in $\tilde X$. Then the elements $e_1, \ldots ,e_l$ form a basis for the $\mathbb Z G$-module $C_k(\tilde X)$ and the kernel of ∂_{k-1} is generated by
$\partial_k(e_1), \ldots ,\partial_k(e_l)$. Remove the k-cells from $\tilde X$ and attach $n-m+l$ free G-orbits of k-cells $Gf_1,\ldots ,Gf_{n-m+l}$ in the following way: suppose that

(α_{ij}), $1\le i \le m$, $1\le j \le n$ is a matrix associated with the epimorphism ϕ. Attach gf_s to $g\sum_{j=1}^m \alpha_{js}\partial_k(e_j)$ for $1\le s\le n$ and attach gf_{n+t} to $g\partial_k(e_{m+t})$ for $1\le t\le l-m$ (we assumed $l \ge m$; if not wedge on an appropriate number of k-balls to X). This yields a new complex $\tilde X_1$. Note that the new boundary map $$\partial_k'=\phi \oplus \partial_k:C_k(\tilde X_1)=\mathbb Z G^n \oplus \mathbb Z G^{l-m}\rightarrow \partial_k(C_k(\tilde X))=\mathbb Z G^m\oplus \mathbb Z G^{l-m}\subseteq C_{k-1}(\tilde X_1)$$ maps the first factor $\mathbb Z G^n$ on the left to the first factor $\mathbb Z G^m$ on the right via ϕ, and the second factor $\mathbb Z G^{l-m}$ on the left to the second factor $\mathbb Z G^{l-m}$ on the right via ∂_k. Hence $H_k(\tilde X_i)=\ker(\phi)=P$. Let X_1 be the orbit complex obtained from $\tilde X_1$ by factoring out the action of G. We have $\pi_k(X_1)=H_k(\tilde X_1)=P$.

We build a second complex X_2 by wedging $n-m$ k-spheres to X. Note that $\chi_d(X_1)=\chi_d(X_2)=n-m+\chi_d(X)$ and $\pi_k(X_2)$ is a free $\mathbb Z G$-module of rank $n-m$. Hence X_1 and X_2 are not homotopy-equivalent. \qed

\bigskip
Let us discuss the case $k=2$. The construction of the complex $\tilde X_1$ works just as well but one should notice that because we are restructuring the 2-skeleton this can have an effect on the fundamental group. In fact, it is possible that the complex $\tilde X_1$ is not simply-connected and the fundamental group of the quotient complex X_1 might be different from G. However we do have two 2-dimensional chain complexes $C_*(\tilde X_1)$ and $C_*(\tilde X_2)$ that have the same directed Euler characteristic but are not chain homotopically equivalent because $H_2(\tilde X_2)$ is free and $H_2(\tilde X_1)=P$, which is not free.

By an {\sl algebraic (G,n)-complex} we mean an exact sequence $${\cal C}: F_n\rightarrow ... \rightarrow F_0 \rightarrow \mathbb Z \rightarrow 0$$ where the F_i, $i=1, \ldots ,n$ are finitely generated free $\mathbb Z G$-modules. If c_i is the rank of the module F_i then the {\sl directed Euler characteristic of ${\cal C}$}, $\chi_d({\cal C})$, is the alternating sum $\sum_{i=0}^n (-1)^{n-i}c_{i}$. Of course, if X is a (G,n)-complex then the cellular chain complex $C_*(\tilde X)$ of the universal covering $\tilde X$ is an algebraic (G,n)-complex.

The above discussion yields the following

\begin{theorem} Let G be a 2-dimensional group and assume P is
a stably-free non-free projective module over the group ring $\mathbb Z
G$ which is the kernel of an epimorphism $\phi: \mathbb Z G^n
\rightarrow \mathbb Z G^m$. Then there are algebraic
$(G,2)$-complexes ${\cal C}_1$ and ${\cal C}_2$ with directed
Euler-characteristic $\chi_{min}(G,2)+n-m$ such that $H_2({\cal
C}_1)$ is isomorphic to P and $H_2({\cal C}_2)$ is free of rank
$n-m$. In particular ${\cal C}_1$ and ${\cal C}_2$ are not
chain-homotopy equivalent.
\end{theorem}

We say a group H is a {\sl retract} of a group G if there are
maps $H\stackrel{j}\rightarrow G \stackrel{p}{\rightarrow} H$ so
that the composition $p\circ j$ is the identity. If M is a
finitely generated $\mathbb Z G$-module, we denote by $d_G(M)$
the rank, that is the minimal number of generators of M.

\begin{lemma} \label{lemma} Suppose H is a retract of G and there exists an
epimorphism $\phi: \mathbb Z H^n \rightarrow \mathbb Z H^m$ with
kernel P and $d_H(P) > n-m$, i.e. P is a stably-free
non-free projective module. Then $\mathbb Z G \otimes_H P$ is a
stably-free non-free projective module over $\mathbb Z G$.
\end{lemma}

\Proof. Clearly the induced module $\mathbb Z G \otimes_H P$ is the
kernel of the induced epimorphism $\mathbb Z G \otimes_H \phi:
\mathbb Z G^n \rightarrow \mathbb Z G^m$. We view P as a $\mathbb
Z G$-module via the epimorphism $p: G\rightarrow H$. Now the
homomorphism $\mathbb Z G \otimes_H P \rightarrow P$ that sends
$g\otimes x$ to $p(g)x$ is an epimorphism. Hence $$d_G(\mathbb Z G
\otimes_H P)\ge d_G(P)=d_H(P)> n-m.$$ Hence $\mathbb Z G \otimes_H P$
is not free. \qed

\begin{theorem} Suppose G is a k-dimensional group, $k\ge 2$,
that contains the trefoil group T as a retract. Then there are
homotopically distinct (G,k)-complexes
with directed Euler characteristic $\chi_{min}(G,k)+1$. In the case where $k=2$
these are algebraic.
\end{theorem}\ldots

\Proof. Let X_1 be the 2-complex of Theorem 2.2. Then $\pi_2(X_1)$
is the kernel of the epimorphism $\phi: \mathbb Z T^2 \rightarrow
\mathbb Z T$ given by $\phi(e_1)= 1+a+a^2$ and $\phi(e_2)=1+b+b^2+b^3$.
Dunwoody shows in \cite{Dunwoody} that
$d_T(\pi_2(X_1))=2$. In particular $\pi_2(X_1)$ is stably-free but

not free. By Lemma \ref{lemma}, $\mathbb Z G \otimes_T \pi_2(X_1)$ is a
stably-free non-free projective over $\mathbb Z G$ that is the
kernel of an epimorphism $\mathbb Z G \otimes_H \phi: \mathbb Z G^2
\rightarrow \mathbb Z G$. The result follows from Theorems 3.1 and
3.2.
\qed

\bigskip\noindent
{\bf Examples.}

\begin{itemize}
\item[(1)] The group $G=T\times \mathbb Z^k$, $k \ge 1$ is
$(k+2)$-dimensional and contains T as a retract. Thus there are
homotopically distinct $(G,k+2)$-complexes with directed
Euler characteristic $\chi_{min}(G,k+2)+1$.

\item[(2)] The group $G=T*\mathbb Z$ is a 2-dimensional group which
contains T as a retract. Thus there are chain homotopically
distinct algebraic $(G,2)$-complexes with Euler characteristic
$\chi_{min}(G,2)+1$.

\item[(3)] Since the commutator subgroup $[T,T]$ of the
trefoil group is free of rank two we see that T is
free-by-cyclic. Indeed, it is not difficult to show that $\langle
x,y,t \mid txt^{-1}=y, tyt^{-1}=x^{-1}y \rangle$ presents T.
Consider the group G presented by $\langle x,y,z_1,\ldots ,z_n,t
\mid txt^{-1}=y, tyt^{-1}=x^{-1}y, tz_it^{-1}=w_i \rangle$, where
$i=1,\ldots, n$ and $w_1, \ldots ,w_n$ is a basis for the free group on the
z_i. Since $G/N=T$ where N is the normal closure of the z_i
we see that G contains T as a retract. Thus there are chain
homotopically distinct algebraic $(G,2)$-complexes with Euler
characteristic $\chi_{min}(G,2)+1$.

\end{itemize}
%\bigskip

We end this section with more comments on the 2-dimensional case.
Suppose X is a standard 2-complex modelled on a presentation for the group $G
$, ${\cal P} =\langle x_1,\ldots ,x_k \mid r_1,\ldots, r_l \rangle$. Then the complex
X_1 of Theorem 3.2 can also be modelled on
a presentation and we will now make this presentation explicit.
First some notation. Let F be the free group on $x_1,\ldots, x_k$,
let R be the normal closure of the relations $r_1, \ldots ,r_l$ in
F and let $p: F\rightarrow G$ be an epimorphism with kernel R.
For every $g\in G$ choose an element $\bar g \in F$ so that
$p(\bar g)=g$. Furthermore choose a total ordering on the

countable set G. If $r\in R$ and $\alpha =\sum_{i=1}^tn_ig_i\in \mathbb Z G$, where $g_1< \ldots <g_t$, then we define $^{\alpha}r=\bar g_1 r^{n_1} \bar g_1^{-1}\ldots\bar g_t r^{n_t} \bar g_t^{-1}$.

Let $\phi: \mathbb Z G^n \rightarrow \mathbb Z G^m$ be an epimorphism and let (α_{ij}) be a matrix for ϕ, $1\le i \le m$, $1\le j \le n$. Define $u_s= {^{\alpha_{1s}}r_1}\ldots^{\alpha_{ms}}r_m$, $s=1, \ldots ,n$. Let ${\cal P}_{\phi}=\langle x_1, \ldots ,x_k \mid u_1,\ldots ,u_n, r_{m+1}, \ldots ,r_{m+(l-m)} \rangle$.

\begin{theorem} Let $\phi: \mathbb Z G^n \rightarrow \mathbb Z G^m$ be an epimorphism, X be an aspherical 2-complex modelled on the presentation ${\cal P} =\langle x_1,\ldots ,x_k \mid r_1,\ldots, r_l \rangle$ for G and X_1 be the 2-complex modelled on the presentation ${\cal P}_{\phi}$. If ${\cal P}_{\phi}$ also presents G then $\pi_2(X_1)$ is isomorphic to the kernel of ϕ. In particular, if the kernel of ϕ is not free of rank $n-m$ then the 2-complexes X_1 and $X_2=X\vee S^2...\vee S^2$ ($n-m$ 2-spheres) are not homotopically equivalent.
\end{theorem}

\Proof Build a 2-complex $\tilde X_1$ from the 1-skeleton of $\tilde X$ and the epimorphism ϕ as in the proof of Theorem 3.2. Note that by construction $H_2(\tilde X_1)$ is the kernel of ϕ. Observe that the orbit complex $\tilde X_1 /G$ is X_1. The assumption that the fundamental group of X_1 is G implies that $\tilde X_1$ is the universal covering of X_1. Hence $\pi_2(X)=H_2(\tilde X)= ker(\phi)$. \qed

\bigskip
In Dunwoody's example the conditions in the theorem are satisfied: The epimorphism $\phi: \mathbb Z T^2 \rightarrow \mathbb Z T$ is given by the matrix (α_1, α_2), where $\alpha_1=1+a+a^2$ and $\alpha_2=1+b+b^2+b^3$. The 2-complex X is modelled on the presentation ${\cal P}=\langle a,b \mid r \rangle$, $r=a^2b^{-3}$. Since the kernel of ϕ is not free of rank 1 and ${\cal P}_{\phi}=\langle a, b \mid {^{\alpha_1}r}, ^{\alpha_2}r \rangle$, $^{\alpha_1}r=r ara^{-1} a^2ra^{-2}$, $^{\alpha_2}r=r brb^{-1} b^2rb^{-2} b^3rb^{-3}$, does present the trefoil group T, the complexes X_1 modelled on ${\cal P}_{\phi}$ and $X_2=X\vee S^2$ are not homotopically equivalent.

\section{An application}

If M is a finitely generated $\mathbb Z G$-module we denote by

$d_G(M)$ the rank of M (that is the minimal number of generators). Let $\cal C$ be an algebraic $(G,1)$ complex. So $\cal C$ is an exact sequence
$${\cal C}: F_1\rightarrow F_0 \rightarrow \mathbb
Z \rightarrow 0$$ where the F_i, $i=0,1$ are finitely generated free $\mathbb Z G$-modules. The module $H_1({\cal C})$ is a generalized relation module for the group G. It has been known for a long time that the difference $d_G(H_1({\cal
C}))-\chi_d({\cal C})$ is an invariant for G in case that G is finite. Dunwoody's exotic presentations show that this result does not extend to finitely presented groups. A natural question is whether similar results hold in higher dimensions. Here is the complete answer for finite groups.

\begin{theorem}(Gruenberg \cite{Gruenberg2}) Let G be a finite group and $\cal C$ be an
algebraic (G,n) complex, $n \ge 1$. Then the difference
$$d_G(H_n({\cal C}))-\chi_d({\cal C})$$
is an invariant of G except when $\mathbb Z G$ fails to allow cancellation and $\mathbb Z$ has projective period $4k$, $k \ge
1$, and $n \equiv 2 (mod \ 4)$.
\end{theorem}

\noindent The exceptional case occurs for example when G is the generalized quaternion group of order 32. Here $\mathbb Z$ has projective period 4 over the group ring $\mathbb Z G$.

\begin{theorem} Let T be the trefoil group and $G=T\times
\mathbb Z^{k-2}$, $k-2\ge 0$. Let $\cal C$ be an algebraic (G,k)-complex. Then the difference
$$d_G(H_k({\cal C}))-{\chi}_d({\cal C})$$
is not independent of the choice of $\cal C$.
\end{theorem}

\Proof. Let X_1 be the 2-complex of Theorem 2.2. Dunwoody shows in \cite{Dunwoody} that $d_T(\pi_2(X_1))=2$. In particular $\pi_2(X_1)$ is stably-free but not free. By Lemma 3.3, $\mathbb Z
G \otimes_T \pi_2(X_1)$ is a stably-free non-free projective over $\mathbb Z G$. The result follows from Theorem 3.2 in case $k-2\ge
1$ and Theorem 3.3 in case $k-2=0$. \qed

\section{Questions and open ends}

Some motivation for the present paper came from the following open question: Can there be homotopically distinct 2-complexes X_1 and X_2 with the same fundamental group G and

Euler characteristic $\chi(X_1)=\chi(X_2)> \chi (G,2)+1$? Dunwoody's examples (and Lustig's generalizations) all have Euler-characteristic exactly one above the minimal level. We believe that our techniques will eventually lead to a positive answer for the above question. The following line of approach seems promising to us.

Let G be a 2-dimensional aspherical group. Choose left module generators $\alpha_1, \ldots ,\alpha_n$ $n\ge 3$, of $\mathbb Z G$. This determines an epimorphism $\phi:\mathbb Z G^n \rightarrow \mathbb Z G$, where $\phi(e_i)=\alpha_i$, $i=1, \ldots ,n$. Suppose that the presentation ${\cal P}_{\phi}$ does define the group G. Let X_1 be the 2-complex modelled on ${\cal P}_{\phi}$. We know from Theorem 3.5 that $\pi_2(X_1)$ is isomorphic to the kernel of ϕ. In order to compute the minimal number of generators for $\pi_2(X_1)$ we choose elements $\beta_i \in \mathbb Z G$, $i=1, \ldots ,n$ so that $\sum_{i=1}^n \beta_i \alpha_i =1$. Then $\pi_2(X_1)$ is isomorphic to $M=\mathbb Z G^n/\beta_1e_1+\ldots+\beta_ne_n$. One can now try to find a quotient of that module for which rank computations can be carried out. If one finds that $d_G(M)> n-1$, then $\pi_2(X_1)$ is not free and hence X_1 is not homotopically equivalent to $X_2=X\vee S^2 \vee \ldots \vee S^2$ (with n 2-spheres added to X).

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