SEMESTER RESEARCH

Archimedes' Discoveries:
A Closer Look

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MTH 467W
Evolution of Mathematics

Contents

Abstract – One of the most notable mathematicians from the culmination of the Hellenistic Age of the Mediterranean, Archimedes, presented numerous discoveries to the world directly and indirectly through his numerous works. This paper will highlight the most essential ideas from his works and put emphasis on their importance to the development of the fields of mathematics and science.

1. Archimedes' Life

a. Life Begins

Archimedes, one of the most notable and famous of the ancient Greek mathematicians, is also one of the most mysterious. Although many of his works have been found, translated, stored in libraries, reproduced, and published, others remain lost. The most information about Archimedes that mathematical historians have today was derived from his surviving works. The mysteriousness of Archimedes dwells within his life outside of his mathematical studies, and what he did with his time other than cry “Eureka!” and draw mathematical figures in the sand.

To better understand the works and discoveries of Archimedes, it is best to start at the beginning of his life history for a good understanding of his background. Archimedes’s birth date and death date are both estimated, although the estimations are more than likely to be nearly accurate. Archimedes lived his entire life in the times of war, and recordings of events were probably kept by higher authorities more accurately around such times. The First Punic War of the Mediterranean started around the year 264 B.C., and Archimedes was born nearly 23 years before that time in approximately 287 B.C. Although that specific year is estimated, it is known for certain that he was born between the years 290 B. C. and 280 B. C. The estimate of his birth year being 287 B.C. arose from the records made by a man by the name of Tzetzes of Byzantine. It has been known (although with some uncertainty) that Tzetzes recorded Archimedes as being 75 years of age at the time of his death in 212 B.C., which puts his birth year at the estimated year 287 B.C. The wars in which Archimedes lived were centered around Syracuse, a once then Greek city-state on the island of Sicily, which happens to be his birthplace.

It has been recorded by Archimedes himself in his work *The Sand-Reckoner* that his father was a worthy astronomer by the name of Phidias (paraphrased Dijsterhuis, 1987). Archimedes is well known for his scientific discoveries as well as his mathematical discoveries. He eventually recognized that the two subjects went hand in hand. His studies in the always-developing field of astronomy at the time were more than likely influenced by his father’s work, or perhaps, their collaborated work.

The historian Plutarch claims that Archimedes and his father were related by blood to the King of Syracuse, Hieron II. It was during the Second Punic war when great friendship can be seen between the King of Syracuse and Archimedes. Because Syracuse was a war-time city, the call of defense arose naturally, and Archimedes beckoned the call with the inventions of his war-time machines. His studies in simple machinery which eventually led to building more complex war machines began in his youth during his travels.

b. Travels

His travels were relatively restricted as for where he went, but they proved to be fruitful nonetheless. His first known expedition was to the city of Alexandria in Egypt. Around 260 B.C., the museum, or university of Alexandria's faculty consisted mainly of Euclid's students. His studies in Alexandria are apparent in his works and often can reflect the personality of Euclid's writing (paraphrased Dijsterhuis, 1987). It has been known that the reason for his journey to Egypt came from an idea from his father, Phidias. Being in a country of constant war, it was probably much better to send your son to study at the university than to have him inducted into battle. If it were not for his education at the Alexandria library, Archimedes may have never been able to cultivate into the great mathematician he became. Archimedean works show that he had great respect and reverence for the faculty of the university. It has been said that Archimedes was particularly fond of the astronomer, scientist, and professor Conon of Samos. It is also said that before Archimedes would publish any of his works, he would run them by Conon for editing and review for content and correctness. Although not many mathematicians dedicated their works to certain people during this time, Archimedes showed his respect and veneration to his professors at Alexandria by doing just that. The dedications included "Erathosthenes of Cyrene for whom the *Method* was written, and Conon's pupil, Dositheus of Pelusium, to whom the works *On the Sphere and Cylinder, On Conoids and Spheroids,* and *On Spirals* are formally dedicated" (Dijsterhuis, 1987). After polishing up his studies and research at Alexandria, Archimedes returned to Syracuse for the remainder of his natural life. Upon returning, he ended up making additional stops to other places. One which is known in historic articles for certain that he made rest at was the country of Spain. During the brief time he spent in Spain, Archimedes aided the Spanish during the English Maritime war. Archimedes expedition to Spain was reported by Leonardo Da Vinci[1]. Other than his trip to Spain, his travels are relatively unknown. It may be possible that he made more than one trip to Alexandria in Egypt.

c. Simple Machines

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[1] Quoted in Favaro, *Archimede*, p.19 (Dijsterhuis, 1987)

The idea of Archimedes as a man of simple machines came from the fact that he created a great amount of unique and oddly useful devices which proved to be worthy and handy in times of war. Archimedes also used his knowledge for creating other machines which made life a little easier. The first reported machine that Archimedes made which commenced the growing ease of laborious life was the Archimedean screw. The device was as simple as it was effective because of its efficiency in operation. The invention of the device was most likely made during his stay in Egypt. The River Nile was used extensively as a water source, and by adding the Archimedean screw to the problem of moving water, fields could be irrigated more easily. Water could then be "harvested" for the townspeople's use rather than by moving it by other means. The Archimedean screw is still used today, over 2,000 years later, in commercial applications and for personal usage for transporting water in an efficient manner around farms and small communities with little water access.

Back in Syracuse, Archimedes got back to life not as a traveler, but as a man of the city. He regained his old friendship back with King Hieron and then began working as a scholar of mathematics, astronomy, physics, and in general, as a scientist.

Archimedes' return proved that war did not cease, but contridictoraly had gotten worse. In an extension of help to King Hieron, Archimedes' used his broad knowledge of physics and mathematics to put together many concepts which would eventually crystalize into the most fantastic ideas. A few of these fantastic ideas may have been designed for detrimental purposes, but they were nonetheless brilliant. These included, but are not limited to, Archimedes' Claw (utilizing a pulley system), Burning Mirrors, a Planetaria, and a ship birthing devices (stating the law of the lever). Archimedes had also made improvements to Leonardo Da Vinci's steam cannon, the ancient catapult system, and Zhang Heng's waterclock.

Archimedes was also known not only for his magnicificent inventions, but for his quirkiness and obsession with geometry and mathematical thinking. Plutarch recorded that during his baths, he would be known (as reported by his slaves) to draw lines and geometrical figures upon his body with oil, as if in a trance (Paraphrased Pickover, 2008). One of the most legendary, and also the most remembered events related to Archimedes is his "bathtub incedent". During the time of the Second Punic War, King Hieron's crown was claimed to have been made impurely by the royal metallurgist. This problem had reached

Archimedes soon enough, and being a man of obsessive thought, he mulled over the problem, not producing an answer for testing this purity of his friend's crown. While preparing for a bath and slipping into the water, he watched the water as it rose while he slid further into the tub. Archimedes then realized what must be done, scampered out of the bathtub, and ran down the streets bare, yelling *evonxa, evonxa!* which translates to *eureka! eureka!* This famous phrase is now coincidentally recongnized with the name "Archimedes". Archimedes ended up creating new theorems for the law of buoyency while he helped solve the mystery of the crown. All of Archimedes' achievements mentioned thus far would not have been possible without working them out on paper with precision and then deriving a final solution in order implement his creation. The creation of new mathematical ideas and theorems is why Archimedes is not only a scientist and one of the worlds most inovative thinkers, but a mathematician as well.

1. Archimedes' Works

Archimedes' love of mathematics resulted in works which splay into a great range of topics including, but not limited to, algebra, trigonometry, ["near"] calculus, geometry, conics, astronomy, number theory, hydrostatics, laws of gravity, laws of motion, and laws of balance.

Archimedes surviving works[2] include:

1. *On the Equilibrium of Planes, Part I*
2. *The Quadrature of the Parabola*
3. *On the Equilibrium of Planes, Part II*
4. *On the Sphere and the Cylinder, Parts I and II*
5. *On Spirals*
6. *On Conoids and Spheroids*
7. *On Floating Bodies*
8. *Measurement of a Circle*
9. *The Sand-Reckoner*
10. *The Method [of Mechanical Theorems]*

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[2] Derived from Bower and Cooke

Two Archimedean works which are not considered to be mainstream nor well studied are the *Palimpsest* and the *Book of Lemmas*. The *Palimpsest,* although it is not a true Archimedean work, is made up of a collection of some of these surviving works as well as a book called the *Ostomachion*, which contains a mathematical puzzle similar to the Tangram puzzle. You are given a set of figures that initially form a square to work with and the puzzle is to make as many animal/object shapes as possible. The puzzle was first derived from Archimedes' studies in number theory and geometrical combinatorics (paraphrased Dunham, 1990).

The *Book of Lemmas*, an apocryphal work of Archimedes, was first written in Arabic and translated by Thabbit ibn Qurra.

The *Book of Lemmas* contains:

1. *The Cattle Problem*
2. *Arbelos*
3. *Salinon*

To best understand how Archimedes affected the evolution of mathematics, it is foremostly important to not only study his history in detail, but his works. His works, even though some have been recopied, lost, misplaced, and recopied again throughout antiquity, they do contain very complex mathematics which did not get simplified over time. These works will now be examined in detail. Perhaps the most important concept to understand when examining such a large set of works is that they tend to be very specific to certain subjects, and some of his problems came from his imagination or from challenges with other mathematicians, making them hard to follow. For example, in the *Sand-Reckoner*, the problem evolved from a combination of both Archimedes' ideas and Aristarchus' ideas on how to figure out exactly how much sand it would take to fill the universe. Some of even the most ludicrous ideas in history eventually proved to be extremely important to the development of a certain area of knowledge (for that specific problem: performing intelligent iterative computations). Archimedes was specifically interested in problems that had a physical meaning tied to them. He preferred to work with anything that was geometric by nature or related in some way to the laws of physics. Since some of the works of Archimedes are long and non-trivial, a summary of each work will be given, and then select problems will be discussed in detail if necessary.

a. *On the Equilibrium of Planes,* Parts I and II

Archimedes is well known for his theories in mechanics and his worked relations between those theories and the actual mathematics that goes into making them. In fact, it is probably safe to say that Archimedes was possibly the first person to relate mechanics and mathematics. Having understood the physical world around him allowed Archimedes to truly delve into hands-on applications that tested his mathematical theories.

On the Equilibrium of Planes is broken up into two parts: I and II. Although it is technically two works, most historians view these two parts as a whole, singular work. It is very common in many texts to not find the name *On the Equilibrium of Planes,* but instead, to find *Centres of Gravity of Planes*. Together, these two equally known titles summarize the entire work perfectly. The work provides theorems on both centers of gravity and equilibriums of planes. Although the two titles sound very similar, they can be separated simply because of the words “planes” and “gravity”. In physics, the study of these two concepts is dealt with differently.

The outline of the work is simple and straightforward. It is easy to say this is true of nearly any Archimedean work. *On the Equilibrium of Planes* is broken up into two main sections: postulates (or assertions) and propositions (or proven statements).

There are in total seven postulates and 10 propositions. The postulates are very simple and easy to understand, but out of their simplicity, Archimedes was able to combine ideas from one or more postulate and create a few very complex views on the theory of equilibrium which are stated in the propositions. To clearly see how Archimedes derived the law of the lever, it is wise to examine the postulates that he used to form it.

Postulates[3]:

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[3] Dijksterhuis, *Archimedes*

1. *We postulate that equal weights at equal distances are in equilibrium, and that equal weights at unequal distances are not in equilibrium, but incline towards the weights which is at the greater distance.*

   This postulate lays the basic theorem for the law of centers and law of gravity in both of the fields of physics and calculus. Archimedes proposed this postulate in a restricted way because of his wording: "equal weights". This is the only postulate that is formed on the basis of equality.

2. *that if, when weights at certain distances are in equilibrium, something be added to one of the weights, they are not in equilibrium, but incline towards that weight to which something has been added.*

   Postulate two simply denotes the addition of a variable force to one end of a balance. This is a key addition that allows for moving forward into deeper examination of planar mechanics. Without variables, Archimedes would not have been able to create any more theorems, nor work with the actual physical objects to which he tested his theories.

3. *similarly that, if anything be taken away from one of the weights, they are no in equilibrium, abut incline towards the weight from which nothing has been taken away.*

   Postulate three takes the idea of adding weights to a balance and negates this idea. By negating the idea, and taking weight away instead of adding it, Archimedes sees the need to negate the result as well so that the balance goes toward the end where weight was not taken away.

4. *when equal and similar figures are made to coincide, their centres of gravity likewise coincide.*

   Archimedes found a very clever way of explaining the linkage between similar sets of objects and their equilibriums by stating that they "coincide". To coincide, or be the same, proves to

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Note*: Postulates are in italics* and examinations are in non-italics.

be a simple concept to grasp: if two sets of objects have the same weights, however different they are, will also have the same centers of gravity and will find equilibrium on a balance.

5. *in figures which are unequal, but similar, the centres of gravity will be similarly situated.*

   Postulate five explains a simple negation concept of postulate 4. If two objects have different weights, their centers of gravity will correspond by not being in equilibrium.

6. *if magnitudes at certain distances be in equilibrium, other [magnitudes] equal to them will also be in equilibrium at the same distances.*

   Archimedes invites a new concept to his postulate group by saying that if two weights are perfectly balanced at one distance, they will continue to be balanced at further (or closer) distances. This postulate only holds if those distances which the weights are from the center are always the same.

7. *in any figure whose perimeter is concave in the same direct the centre of gravity must be within the figure.*

   To best understand postulate seven, it is easiest to imagine the concept in a 3D view. If the "surface area" contained by the perimeter of the objects is concave (bends in any way), then the center of gravity must follow similarly by remaining within the perimeter of the figures.

Each postulate provides a simple concept for any variations within a plane filled with weighted objects. In the propositions in the second section of *On the Equilibrium of Planes*, Archimedes takes ideas from one or more postulates and creates statements which can be easily proven or disproven. The most important fact to observe is that the propositions, together, form the Law of the Lever. Before Archimedes had initially studied and provided work on his law of the lever, Aristotle had first formulated his version of the law. Most mathematical historians will state that Archimedes' version is more widely accepted over Aristotle's vague and imprecisely made propositions on equilibriums of planar objects (Fauvel, et al., 1987). Archimedes's view on equilibrium was more soundly based because it followed a static nature that is explained only by the concept of equilibrium. Aristotle had based his formulations from the kinetic laws of motions which limited its power to represent any object in a plane that had a dynamic weight (paraphrased Bower, 1989).

Archimedes had cleverly adopted a more modern version of Aristotle's principles of equilibrium and he had birthed one of the most basic and important laws of physics. In modern physics, the law of the lever transforms into the law of torque which can be derived from studying weights, momentum, and applications of Archimedean postulates.

The modern law of the lever states:

> Given two objects with weights, α_1 and $\propto_2$, sitting two distances, β_1 and β_2, from a center [fulcrum] it must be true that $\alpha_1 \times \beta_1 = \alpha_2 \times \beta_2$. This gives the equilibriums[4]:

$$\beta_2 = \frac{\alpha_1 \beta_1}{\alpha_2}, \quad \beta_1 = \frac{\alpha_2 \beta_2}{\alpha_1}, \quad \alpha_2 = \frac{\alpha_1 \beta_1}{\beta_2}, \quad \text{and} \quad \alpha_1 = \frac{\alpha_2 \beta_2}{\beta_1}.$$

Archimedes had used this principle to impress his friend Hieron II. Archimedes was a novice and had just got his feet on the ground by formulating theorems for his simple machines when Heiron II had wanted to truly test Archimedes' ability. Hieron II had picked out one of his largest undocked ships and asked Archimedes to birth it by himself. Around the same time that Archimedes had contrived his equilibrium laws, he was working with pulley systems. Archimedes knew exactly what do and successfully pulled the ship out to sea with a cunning pulley structure he devised. King Hieron II was so impressed with Archimedes' work that he said "from this day forth Archimedes is to be believed in everything he may say"[5]. Archimedes had become so confident in his abilities because of his success that he stated that there are no limits to ideal mechanics that can be proven with mathematics, and he then stated his famous quote: "Give me a place to stand on, and I can move the Earth" (Cooke, 2005).

b. The Quadrature of the Parabola

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[4] Word Mathematics Add-in

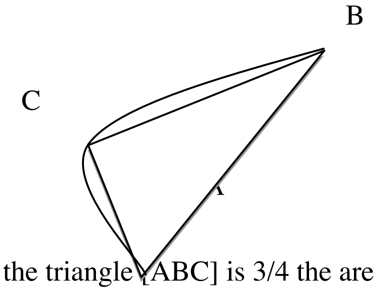
[5] Dialogue from *The Sand-Reckoner* by Bradshaw

Archimedes first proved his expertise in geometrical studies in *The Quadrature of the Parabola*. Before the work was published it was actually sent in full form to Archimedes' acquaintance, Dositheus. Although the work contains over 20 propositions that cover the complexities of parabolas and their qualities, there exists only one of those propositions that stand out as the most significant. Archimedes had proved this one proposition by use of geometry and also by way of mechanical evaluation.

The proposition in question is the 24th in the book. It states[6]:

> *Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height.*

To imagine the figure described by Archimedes is simple if the proposition is well read. In layman's terms, Archimedes is simply stating that if you have a parabola, with sides A and B, then C should be a point that is tangent to the chord of the parabola [AB].



By geometrical proof, Archimedes explains that the area of the triangle [ABC] is 3/4 the area of the parabola, or subsequently, that the area of the parabola is 4/3 times the area of the triangle [ABC]. Archimedes went through a great depth of thought to create a geometric proof for this proposition. He first used two additional tangents on sides [AC] and [BC] to help construct a parallelogram around [ABC] with the parallel sides being the tangents of the arc [C]. By calculating different areas within the triangle, the parabola, and the parallelogram, Archimedes was finally able to formally deduce that his postulate was indeed true. One thing to note from *On the Quadrature of the Parabola* is that Archimedes used the concept of area many times during proofs of his theorems. Stein explains the importance of this by saying, "not until the end of the nineteenth century did mathematicians bother to define area and consider such questions [concerning the area]" (Stein, 1999). Showing that Archimedes used lesser known ways to prove his postulates indicate that he was ahead of his time and also that he was one of the forbearers for motivating the evolution of mathematics during his age.

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[6] Heath, 1897

The examination of postulate 24 shows realization that Archimedes was touching the boundaries of an unknown subject during his time: calculus. By geometric proof, Archimedes came across an infinite series, although he didn't exactly call it this at the time.

By observation of proposition 24, Archimedes shows:

Parabolic area = $\frac{4}{3}\,(\Delta\, ABC)$

By the method of exhaustion by, Archimedes finally arrived at what is known today as an infinite series in order to find the area ratio of the parabola and the triangle by inscribing sequences of polygons inside one another until they eventually converge to some $L \ni (-\varepsilon, +\varepsilon)$[7]. This gives:

$$\left(1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\ldots+\frac{1}{4^n}\right)\times(\Delta\, ABC)$$

Thus leading to the geometric series calculation:

$$\left(\sum_{i=1}^{n}\frac{1}{4^i}\right)\times(\Delta\, ABC)$$

Archimedes lived hundreds of years before both Leibniz and Newton, yet he is the first to be called by many mathematical historians, "the father of integral calculus". His work *On Spirals* will show this more even more so than *On the Quadrature of the Parabola*.

c. *On the Sphere and the Cylinder,* Parts I and II

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[7] Convergence theorem of Calculus II

On the Sphere and Cylinder proves to be one of Archimedes' most accomplished works. Archimedes himself believed that two distinct propositions which dwelled within this work, together, made up what was his greatest accomplishment. They are the finding of the ratios of both the internal area and surface area of the sphere and cylinder. This work is the second that we examine that was also a letter to his friend Dositheus. The work contains introductory propositions and postulates describing various details and geometric analysis of spheres, cylinders, and cones. Propositions 33 and 34 from Part II of *On the Sphere and* Cylinder are the ones noted above.

Proposition 33 states[8]:

The surface of any sphere is equal to four times that of its greatest circle.

Proposition 34 states[9]:

Any sphere is equal to four times the cone whose base is equal to the greatest circle of the sphere, and whose height is equal to the radius of the sphere.

Archimedes thus derived the following:

If (height(cylinder) = diameter(sphere)) then:

Volume(cylinder) = 3/2(volume(sphere))

To put this into perspective[10]:

[As provided by Archimedes' assumption of the ratio]

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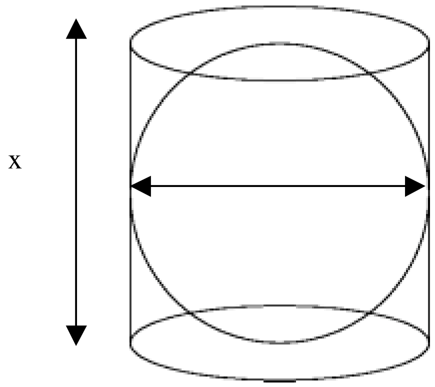
[8] Allen

[9] Allen

[10] Example taken from chapter 10, 11 homework, problem 1.c (Archimedes and Apollonius) [Image: Word insert]



Volume(cylinder) = $\pi r^2 h = \frac{1}{4}\pi d^2 h, where\ d, h = x$

Volume(sphere) = $\frac{4}{3}\pi r^3 = \frac{\pi d^3}{6}, where\ d = x$

Assume that these two volumes are equal.

$$\frac{1}{4}\pi x^2 x = \frac{\pi x^3}{6}$$

Multiple throughout.

$$\Rightarrow \frac{1}{4}\pi x^3 = \frac{\pi x^3}{6}$$

Multiply both sides by the value of six from the right side.

$$\Rightarrow \frac{6}{4}\pi x^3 = \pi x^3$$

Reduce.

$$\Rightarrow \frac{3}{2}\pi x^3 = \pi x^3, yielding\ the\ difference\ \frac{3}{2}$$

The difference of 3/2 appears from the last equation, but Archimedes really said that the difference in the ratios was 2/3. To get to this, the ratio must be switched and it now becomes that the sphere has 2/3 the volume of the cylinder by doing the following:

$$4 \times \frac{1}{4}\pi x^3 = \frac{4\pi x^3}{6}$$

Reduce.

$$\pi x^3 = \frac{2}{3}x^3$$

This yields the difference of 2/3 which Archimedes had originally explained in the propositions.

d. *On Spirals*

One of Archimedes' most calculus-like works, *On Spirals,* provides an example on how he solved one of the three most famous problems of mathematical antiquity: squaring the circle. He used a spiral, also known as the Archimedean Spiral, imposed on a plane to solve this problem. Archimedes describes the spiral in his introduction note[11]:

> *A straight line in the plane revolves at a constant rate about one of its ends (which remains fixed) and returns to its starting position. At the same time a point moves at a constant rate along the moving line beginning at the fixed in end and describes a spiral.*

The Archimedean Spiral[12] showing angle trisection:

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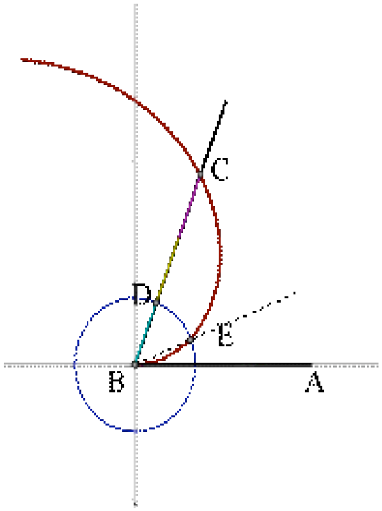
[11] Stein, 1999

[12] Image from PBroks13, editor.



To trisect ABC, the Archimedean Spiral is used. BD is known to be one third BC. From the circle with center B and radius BD, supposed that the center intersects the spiral at E (the tangent). This leads to the fact that ABE is 1/3 ABC, giving the trisection of the angle.

What Archimedes describes in this introduction provides as a precursor to polar coordinates. In fact, Archimedes is said to be one of the first people to describe the coordinates (*r, theta)* in the polar plane.

e. On Conoids and Spheroids

On Conoids and Spheroids proves to be yet another set of propositions and postulates sent to Dositheus. It is a substantial work consisting of over 30 propositions. The most important of these evaluates the areas of solids of revolution. Although Archimedes did not use calculus to integrate these functions to get the area within, he did use the method of exhaustion which is the precursor to formal calculus. These important propositions are[13]:

Parabola revolutions: Propositions 21 and 22.
Hyperbola revolutions: Propositions 25, 26.
Spherical revolutions: Propositions 27 through 32.

Although *On Conoids and Spheroids* is one of the lesser studied works of Archimedes, it proves to be one of the most important. Many mathematicians have dwelled over this work and in turn created great theorems and formulas for the field of calculus. Such ideas include the Disc and Cylinder methods for integrating the areas between the curvatures of a solid of revolution.

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[13] Cooke, 2005

f. On Floating Bodies, Parts I and II

Like *On the Equilibrium of Planes and On the Sphere and Cylinder*, *On Floating Bodies* is a work made of two books: part I and part II. This work was the brainchild of the gold crown solution. When Archimedes had found a solution to that problem, he dove deeper into the subject of physics and water and he in turn created the field of hydrostatics. Hydrostatics is the study of the physics of water that is at rest, but under pressure.

One of the most important ideas that are in *On Floating Bodies* is the formula for finding force of motion due to density differences within different fluids contained in a body.

The equation[14] is:

$$\mathrm{Ar} = \frac{g L^3 \rho_\ell (\rho - \rho_\ell)}{\mu^2},$$

where g is gravitational acceleration, _l is density of the fluid, _ is density of the body, _ is dynamic viscosity, and L is length of the body.

Archimedes was well known for taking a mix of everything he was studying to push the boundaries on what he could solve. For example, he would take paraboloidal (parabola solid of revolution) figures and study their motions in a body of water.

Without *On Floating Bodies,* we would not have had hydrostatic mathematical contributions for many years. Simply because Archimedes found interest in bathtubs and ships, he created these postulates and inspired many others to continue studying the subject and make improvements to theorems for working bodies in water and their mathematical physics.

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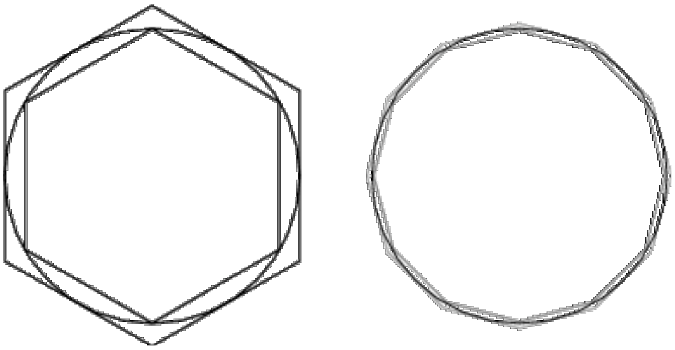
[14] Tobias, 1981

g. *Measurement of a Circle*

Measurement of a Circle is a rather short work. Many people refer to this work as a treatise instead of a formal full-bodied text (paraphrased Dunham, 1990). Even though Archimedes was becoming more in depth with three dimensional objects like solids of revolution, he was still working with two dimensional figures like circles and triangles. Many mathematicians throughout antiquity have attempted to find the true value of the constant Pi, and during Archimedes' time, he had successfully made the most accurate approximation. By using the method of exhaustion and inscribing increasingly-sided polygons within and outside a circle, he would get closer and closer to the true value of Pi found within the circle's property ratios. It is important to note that Archimedes did not think of the value as "pi", but as the property ratio of circumference to diameter.



Archimedes' method of exhaustion to find the ratio known as pi.

[15]

After 96 iterations, Archimedes unsurprisingly was looking at two 96-gon figures with a circle inscribed between them. He ended up finding that the ratio is between to irrational numbers: 3.14103 and 3.14271. More appropriately:

$$3\frac{1}{7} > \pi > 3\frac{10}{71}$$

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[15] From EUREKA!.pptx shown in class.

Although this is not that amazing of an answer (powerful computers have found pi with a 15 billion decimal approximation!), it was exceptional for his time.

h. The Sand-Reckoner

Although *The Sand-Reckoner* contains only one problem, it is no less complex than Archimedes' other works. This work is one that provokes thought and interest to the casual reader, but for the picky logician or mathematician, it can be quite a laborious read to mull over. The work came about when Archimedes was discussing a particular problem with his friend Aristarchus. During Archimedes' time, the Greek numbering system was transitioning from Attic numeration to Ionian numeration (paraphrased Boyer, 1989). Many mathematical historians believe it was because of this fact that Archimedes was able to more easily explain the problem of the *Sand-Reckoner* because larger numerical values were easier to write in Ionian than in Attic numeration.

The problem that was stated was from Archimedes to Gelon, the son of Hieron II[16]:

> *Some think that the number of grains of sand is infinite. I will try to show by means of geometrical proofs that any number can be expressed which exceeds the number of grains of sand on the Earth, which, as stated has a volume equal to that of the earth filled up in the way with such grains of sand, and also of the sand which has a volume equal to that of the Cosmos. The Cosmos is hereby stated by astronomers as the sphere whose centre is the centre of the Earth and whose radius is equal to the distance between the center of the sun and the center of the Earth. We shall now find this number in question.*

In order for Archimedes and Aristarchus to make such a calculation possible, they needed to find a few estimated values for the distance between certain heavenly bodies in order to find the dimensions of the Cosmos. Archimedes made the assumptions based off of Aristarchus's calculations of the distances. When describing the numbers that correspond to the assumptions, Archimedes found it convenient to give

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[16] Paraphrased from Cooke, 2005

a naming convention for certain amounts. He ended up calling these values myriads. The myriad is also called the order of numbers to the fourth power, or more appropriately, $10^{4.}$ A myriad myriad simply denotes that the value is doubled, giving 10^{8} . The first assumption that Archimedes made is that the perimeter of the Earth is three hundred myriads of stadia, estimated to be about 30,000 miles[17]. Archimedes then iterates and finds that value for the diameter of the Earth is greater than the diameter of the moon and that the diameter of the sun is greater than the diameter of the Earth, yielding the fact that the Sun's diameter is many times greater than that of the moon (paraphrased Dijksterhuis, 1987). Archimedes finally arrived at a size for the Cosmos by saying its diameter is 100,000 miles.

Now that Archimedes had an approximation for the Cosmos, he just needed an approximation for the size of a grain of sand and the problem would be solved. In order to find the size of a grain of sand, he said that 10,000 grains of sand is larger than a poppy seed, iterating from the poppy seed saying that the seed is greater than 1/40 of a finger width, and that a stadium is greater than 10,000 seeds (paraphrased Boyer, 1989). By dividing the size of the Cosmos by the size of the grain of sand, Archimedes said that the number of grains of sand that it would take to fill the universe is 10^{51} .

Although this problem and it's solution may not have be very accurate or plausible, it has provided insight to mathematicians who have studied it. It is a great example that shows the power of iterating through a problem one step at a time in a computation manner to arrive at an answer.

i. *The Method* [*of Mechanical Theorems*]

The *Method,* or *The Method of Mechanical Theorems*, was a work that was originally sent as a treatise to Eratosthenes. It was considered a lost work originally, but it was found in the year 1906 at Constantinople (Boyer, 1989). The *Method* can be thought of as a collection of Archimedes' most impressive proofs, including his favorite proof describing the ratio between cylindrical and spherical measurements. It includes propositions from *Quadrature of the Parabola, On Spirals, On the Sphere and Cylinder, Measurement of a Circle, On Floating Bodies,* and *On the Equilibrium of Planes*. In the "greeting" section of the work, Archimedes writes to Eratosthenes that he was motivated to send the work

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[17] Boyer, 1989 data

simply because he thought Eratosthenes would enjoy it, being a "diligent and excellent teacher of philosophy" (*The Method of Mechanical Theorems*, introduction).

When *The Method* was discovered in 1906, it was thought to have contained mathematical content even though it was a washed out copy. If it were not for the eager experts recovering works like those at Constantinople, many of the mathematical works that the world has access to today would simply not exist.

j. The *Book of Lemmas*

The *Book of Lemmas*, considered to be "miscellaneous" works from Archimedes, is also known to be an apocryphal treatise. It is considered an apocryphal because in the text, there is a reference to Archimedes himself, and it is commonly thought that Archimedes would never reference himself in his own work. It was only recently known that this book was attributed to Archimedes from the Arabic mathematician, Thabit Ibn Qurra. The work contains two geometrical figures, the Arbelos and Salinon, which translate to "shoemaker's knife" and "salt-cellar" because of their distinct shapes. Each of these figures is very delicate and many geometrical calculations can be used with them. The only non-geometrical work included in *The Book of Lemmas* is called *The Cattle Problem.*

k. *The Cattle Problem*

The Cattle Problem proves to be yet another computational iterative problem that eventually led to what we know today as the Pell equation. The Cattle Problem stated by Archimedes's apocryphal work is laid out as a challenge to the reader, stating[18]:

> *Oh Stranger, computer the number of cattle of the Sun, who once grazed upon the fields of the Isle of Sicily, divided into four Herds of different colors – milk white, glossy black, yellow, and dappled.*

The text continues and explains that the bulls were divided into the following equations:

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[18] Dijksterhuis, 1987

White bulls = (1/2 + 1/3) Black bulls + yellow bulls.

Black bulls = (1/4 + 1/5) Dappled bulls + yellow bulls.

Dappled bulls = (1/6 + 1/7) White bulls + yellow bulls.

White cows = (1/3 + 1/4) Dappled herd.

Yellow cows = (1/5 + 1/6) White herd.

The problem challenges the reader to find the total number of cattle under the sun using the equations above. There is a catch to the problem though; the *Cattle Problem* puts a constraint on the problem by saying:

White bulls graze together with the Black bulls in rows, such that the number of cattle in each row is equal and that the number was each to the total number of rows, forming a perfect square [matrix]. Yellow bulls graze together with Dappled bulls, but they are formed as a triangle with a single bull in the first row, two in the second, and so on.

The problem states that if the reader is able to find the number of cattle, they are to be deemed with "a crown of glory and knowledge that they are judged perfect in the species of wisdom" (Pickover, 2008). The problem is worked out through brute force iterations, and it wasn't successfully solved until 1880. The first computer to solve the problem was the IBM 7040 series architecture from the year 1965. It is important to note that the problem was stated in the third century B.C. and it remained unsolved until the 17th century A.D. That is a difference of 22 centuries where the problem remained unsolved. Although there are more than likely some mathematical problems that exist that have not been solved yet, it is very rare to see a problem to be unsolved for such a long period of time.

3. Archimedes' Death

Archimedes, one of the last of the Greek mathematicians, died in an unfortunate way. During the fall of Syracuse during the Second Punic War, Archimedes was drawing geometrical figures in the sand like he was accustomed to when a soldier approached him, asking him to stop immediately. Highly disturbed by the soldier, Archimedes refused to obey and continued studying the objects he was drawing and he sputtered his famous last words “*Noli turbare circulos meos*!“, which translates to *“Do not disturb my circles!”*. After a second warning, the soldier killed Archimedes even though he had order from the Roman ruler Marcellus to not harm the mathematician.

Archimedes is without a doubt one of the most talented and interesting of the ancient Greek mathematicians. The thing that made Archimedes so interesting is that he put his mathematical studies into actions through the inventions of his numerous devices. Archimedes’ discoveries have influenced a great number of mathematicians after his time, including Leibniz, Newton, and Diocles. Archimedes is often known as the “father of integral calculus” for his related ideas even though the modern concept of calculus was not invented until well after his time. Archimedes is also known as “the father of mathematical physics” by linking the fields of mathematics and mechanics together. He was a man of a very broad spectrum of expertise, making him one of the greatest scientists to date. Without Archimedes in the picture of the evolution of mathematics, the field would not have developed to what it is today in the time frame that it did. Archimedes set the high bar for the amount of mechanical and mathematical discoveries made during his age and he has kept it there for many years. It will be a wondrous day when that bar is overtaken by new, unthought-of discoveries from mathematicians to come in the future in the continuing evolution of not only mathematics, but all fields of knowledge.

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