NEWTON, LEIBNIZ AND THE CALCULUS

Abstract. The calculus has an outstanding role in mathematics, science and engineering. The work on infinite processes has a long history going back to Greek mathematicians, however Isaac Newton and Gottfried Wilhelm Leibniz have to be credited with the development of the generalized concepts. Their approaches to calculus varied substantially due to the different academic backgrounds of both mathematicians. Both Newton's dynamic approach and Leibniz' static approach are coexisting nowadays and are both of importance in their respective areas of application. The fierce controversy about the priority of merit that followed the publication of the results could not alter this fact in the least. Newton and Leibniz both have contributed much more to science than the calculus, being Newton's discovery of the nature of colors and Leibniz' study of the binary system only two examples.

1. Introduction

Many texts describe Newton and Leibniz as being the inventors of calculus. There are others (e.g. Bressoud, 2009) who think that this description might result in an underestimation of the work of others who contributed to the subject before. The evolution of subjects in mathematics is a process rather than a collection of isolated events. However, for exponents of either point of view there is no doubt about the immense impact that Newton's and Leibniz' works had on mathematics and science. But it is equally important to recognize the contributions of others to calculus before their time. Newton was quoted with the statement: "If I have seen farther than Descartes, it is because I have stood on the shoulders of giants." (Boyer, 1991). There is probably no better way to give credit to those who paved the way for our two protagonists.

The calculus is concerned with the study of infinite processes, be it infinitely small or infinitely large. The step from the finite to the infinite is a tough challenge to imagination and a quote from Jean Le Rond d'Alembert, French mathematician from the 18th century shows the dilemma. D'Alembert apparently was not willing to accept the continuity of the

step when he writes that “a quantity is something or nothing; if it is something, it has not yet vanished; if it is nothing, it has literally vanished.” He went on to state that accepting an intermediate state between these two was a “chimera” (Boyer, 1991).

It was Zeno of Elea in the 5th century BC who was one of the first to challenge his colleagues with the idea of the infinite. He constructed mental experiments referred to as “Zeno’s paradoxes” where by means of repeated fractioning he eventually obtains an infinite subdivision of finite intervals in time or space. He then draws conclusions concerning the infinitely small that are typical for thinking about finite values and obtains paradox statements (Boyer, 1991).

Zeno’s Greek successor, the mathematician Eudoxus of Cnidus, developed the method of exhaustion which is based on the concept of continuity if a quantity is repeatedly increased or decreased. This method has been extensively used for centuries to prove a variety of theorems. The concept finds a straightforward expression in the Archimedean principle that for each number $x > 0$, there exists a positive integer n such that $1/n < x$ (e.g. Dangello and Syfried, 2000).

In the 12th century, the Indian mathematician and astronomer Bhāskara used the concept of infinitely small differences between values in space or time. He stated what is now known as “Rolle’s theorem” which allowed him to find extrema of a curve and derivatives for some specific functions (Anonymous, 2009b).

The idea of the infinitely large is inherent in the construction of infinite series which are the sum of an infinite number of terms. It was equally in India where in the 15th century such series were first used by Mādhava of Sangamagrama to approximate some trigonometric functions (Anonymous, 2009c). Only a century later in Europe, James Gregory worked on infinite series expansions of functions and infinite processes in general (Boyer, 1991). He

rediscovered a theorem that had been stated by Mādhava of Sangamagrama, the arctangent series for θ : $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - + \cdots$.

It becomes clear that a considerable amount of work had been done in relationship with the concept that underlies the calculus, the infinite processes. These efforts, however, had not been integrated into a comprehensive theory but consisted of a collection of special instances of the general concepts. It needed the genius of Isaac Newton and Gottfried Wilhelm Leibniz who in the 17th century constructed this general theory referred to as the calculus.

2. The Calculus

Calculus is the mathematical science that studies infinite processes and the mathematical constructs that carry per se the concept of the infinitely small or large are sequences and series. Both are essential basic tools of the differential and integral calculus developed by Isaac Newton and G.W. Leibniz. It is no surprise that both these mathematicians started their journey into the calculus by studying infinite sequences and series.

Zeno's paradoxon of the dichotomy illustrates the construction of both, the infinite sequence and series. In his mental experiment he states that before one can go a certain distance, one first has to go half that distance, but also half of that distance, and before that half of the latter distance, and so on. Each of these distances is one term of the following infinite sequence:

$$\{\cdots, \frac{1}{2^n}, \cdots, \frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\}$$

where the full distance is denoted by 1 and each term to the left is a fractional part of this total distance. Zeno takes this idea one step further and claims that when these distances become smaller and smaller, the initial distance is ultimately zero, hence the motion does actually never start at all. Using modern notation introduced by the German mathematician Karl Weierstrass in the 19th century (Estep, 2005), this initial distance is the limit of the

above sequence when n gets infinitely large, i.e. when one halves the distance an infinite number of times:

$$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

An alternative way of describing the situation is to say that we first have to move half of the distance, then half of what is left to go towards the end point, then again half of the remainder, and so on. In mathematical terms, we add the terms of the above sequence, starting with $\frac{1}{2}$ and obtain an infinite series of terms:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

Will we ever be able to go the full distance? The answer is yes, but we would have to go half the remaining distance an infinite number of times:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Zeno's paradoxon of the dichotomy is an excellent example of the concept of the limit.

G.W. Leibniz was specifically fond of the arithmetic and harmonic triangles (Boyer, 1991).

Arithmetic or "Pascal" triangle

1	1	1	1	1	1	1	⋯
1	2	3	4	5	6	⋯	
1	3	6	10	15	⋯		
1	4	10	20	⋯			
1	5	15	⋯				
1	6	⋯					
1	⋯						

Harmonic triangle

1/1	1/2	1/3	1/4	1/5	1/6	⋯
1/2	1/6	1/12	1/20	1/30	⋯	
1/3	1/12	1/30	1/60	⋯		
1/4	1/20	1/60	⋯			
1/5	1/30	⋯				
1/6	⋯					

Both these triangles are built on infinite sequences in each of their rows, columns and diagonals. Furthermore, if all terms in one row of the harmonic triangle are added, one obtains infinite series for each of these rows. The first row is the divergent harmonic series, but all other rows form convergent series.

The calculus is mainly concerned with two operations on functions, differentiation and integration. Infinite sequences and series represent the essential tool to successfully handle both of these. As one prior step towards his calculus, Isaac Newton studied the binomial theorem which in his traditional form was nothing else but the combination of integral powers of finite polynomials and Pascal's triangle, where the latter yield the coefficients of the individual terms once the polynomials were multiplied out. One of Newton's first discoveries was the extension of the binomial theorem from integers to rational powers which enabled him to make the step from finite to infinite power series of terms in one variable (Boyer, 1991).

Differentiation is the operation whose result describes the rate of change of a parameter with respect to another. It therefore connects two parameters in a functional relationship whenever one of these parameters depends on the change of the other. Examples are speed as change of distance in terms of time: speed $\propto \frac{\mathrm{d}}{\mathrm{d}t}$(distance) or heat flow as change of temperature in terms of distance: heat flux $\propto \frac{\mathrm{d}}{\mathrm{d}x}$(temperature).

The process of differentiation using infinite series is illustrated by the example of the function

$$y = x^{\frac{3}{2}} \qquad \Rightarrow \qquad y^2 = x^3$$

Since we're interested in the rate of change of the curve near any point (x, y) on the curve, we move to another point $(x + p, y + q)$ in the immediate neighborhood of (x, y). To make sure that this point is very close to the original one, Newton multiplied the changes of the two coordinates by a very small quantity, which he called o. The equation has now the form:

$$(y + oq)^2 = (x + op)^3$$

We then use the binomial theorem and rewrite both sides of the equation as a sum of terms. We obtain:

$$y^2 + 2y(oq) + (oq)^2 = x^3 + 3x^2(op) + 3x(op)^2 + (op)^3$$

Note that we obtain two finite sums on both sides. As mentioned earlier, Newton had generalized the binomial theorem to all rational exponents, therefore the squaring of the equation would not have been necessary and we would have obtained an infinite series for $x^{\frac{3}{2}}$ instead. However, as can easily be seen from the following steps, this wouldn't have changed the overall procedure.

The next and crucial two steps are (1) to divide through by our small quantity o and then to (2) disregard all terms that still contain o:

$$\frac{y^2}{o} + 2y \cdot q + oq^2 = \frac{x^3}{o} + 3x^2 \cdot p + 3xop^2 + o^2p^3 \tag{1}$$

$$2y \cdot q = 3x^2 \cdot p \tag{2}$$

Since our neighborhood around our point (x, y) was very small, we can consider the small piece of the curve between (x, y) and $(x + op, y + oq)$ to be a line, and its slope is the ratio $\frac{q}{p}$. We can use equation (2) to write:

$$\frac{q}{p} = \frac{3x^2}{2y}$$

From our original equation we know that $y = x^{\frac{3}{2}}$, so we obtain:

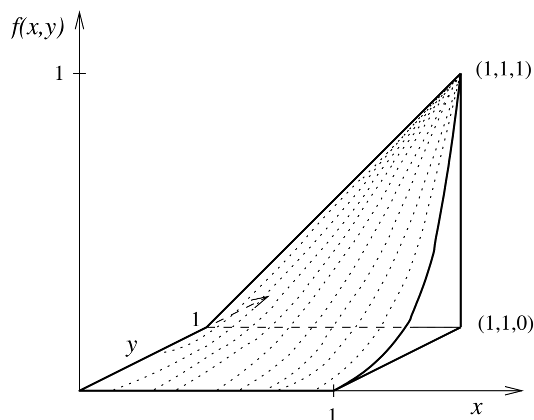
$$\frac{q}{p} = \frac{3}{2}\frac{x^2}{x^{3/2}} = \frac{3}{2}x^{(2-\frac{3}{2})} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

The equivalent written in our usual notation is of course:

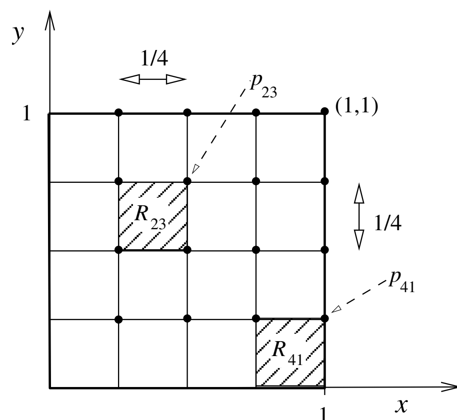
$$f(x) = x^{\frac{3}{2}} \quad \text{and its derivative:} \quad \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{3}{2}x^{\frac{1}{2}}$$

We will consider an example of an integration to illustrate how the use of the concept of the infinitely small allows for an exact evaluation of an integral instead of an approximation.

Let $f(x, y) = xy^2$ be a function defined on the unit square $0 \leq x \leq 1$ and $0 \leq y \leq 1$. We want to find the volume of the body limited below by the unit square and above by the function xy^2. The following figure illustrates:



We can visualize the volume as the sum of columns with height $f(x, y) = xy^2$, so we subdivide the unit square in small squares. Intuitively we know that the finer the subdivision of the base, the better the evaluation of the volume will be compared with the exact volume. We can start with the subdivision as shown in the next figure.



This subdivision is of course far too coarse, but it helps to visualize what we intend to do. We'll make the grid much finer once we've set up the equations.

We have sudivided the square in 4^2 subsquares with side length of $1/4$ each. Furthermore, we've chosen one point representing each of these squares, the point in the upper-right corner. The coordinates of each of these points are $(i/4, j/4)$, such that the point p_{41} which represents the subsquare R_{41} has coordinates $p_{41} = (1/4, 1)$ and the point p_{23} which represents subsquare R_{23} has coordinates $p_{23} = (1/2, 3/4)$.

It is now straightforward to set up all this for the general case where we use a subdivision into k^2 subsquares with side lengths of $1/k$. A point p_{ij} which represents subsquare R_{ij} has then the coordinates $p_{ij} = (i/4, j/4)$ (e.g. Buck, 1978).

We're now ready to add up all our columns. We call this volume V_k because it depends on the parameter k. We obtain

$$V_k = \sum_{i,j=1}^{k} f(p_{ij}) \cdot R_{ij}$$

If we use our point coordinates for the function $f(x, y) = xy^2$ and remember that each subsquare has area $1/k^2$, we obtain:

$$V_k = \sum_{i,j=1}^{k} \left(\frac{i}{k}\right)\left(\frac{j}{k}\right)^2\left(\frac{1}{k^2}\right)$$

We can take k^5 out of the sum since while adding up all subsquares, k remains unchanged. We also observe that both indices take values from 1 to k, so we have a total of k^2 terms for our entire sum. We can represent these facts by the following equation:

$$V_k = \frac{1}{k^5} \sum_{i=1}^{k} i \sum_{i=1}^{k} j^2$$

The first of the two sums is the sum of all integers from 1 to k and thanks to the Pythagoreans (Boyer, 1991) we know that this sum is a triangular number:

$$\sum_{i=1}^{k} i = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

The second of the two sums is the sum of all squares from 1 to k^2 and Archimedes showed us the following identity for this case:

$$\sum_{i=1}^{k} i = 1 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Here is the Greek way to express our volume:

$$V_k = \frac{1}{k^5} \cdot \frac{k(k+1)}{2} \cdot \frac{k(k+1)(2k+1)}{6}$$

We multiply everything out on the numerator and divide the result by k^5 to obtain:

$$\begin{aligned} V_k &= \frac{1}{k^5} \cdot \frac{k(k+1)}{2} \cdot \frac{k(k+1)(2k+1)}{6} \\ &= \frac{2k^5 + 5k^4 + 4k^3 + k^2}{12k^5} = \frac{2 + \frac{5}{k} + \frac{4}{k^2} + \frac{1}{k^3}}{12} \end{aligned}$$

We did the last step for a reason, of course. We said at the beginning, that our subdivision into k^2 subsquares would be too coarse and we wanted a much finer subdivision. We can certainly achieve this if we let k get bigger and bigger, the ideal case would be to have k be infinite. We can do this:

$$V = \lim_{k\to\infty} V_k = \lim_{k\to\infty} \frac{2 + \frac{5}{k} + \frac{4}{k^2} + \frac{1}{k^3}}{12} = \frac{2 + \lim_{k\to\infty}(\frac{5}{k} + \frac{4}{k^2} + \frac{1}{k^3})}{12} = \frac{1}{6}$$

In our modern notation (which by the way was introduced by Leibniz!) we would write:

$$V = \int_0^1 \int_0^1 xy^2 \mathrm{d}x\mathrm{d}y = \int_0^1 \left[\frac{1}{3}xy^3\right]_0^1 \mathrm{d}x = \frac{1}{3}\int_0^1 x\mathrm{d}x = \frac{1}{3}\left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{6}$$

Infinite processes are involved in all of the procedures briefly sketched in this section. There are cases when simple ratios or multiplications are sufficient to differentiate or to integrate. This is true if the functions that are treated are constant or change only uniformly. Both these cases are only special cases of the general situation. The instances where mathematicians had found tangents to curves or integrals under curves in the past were mostly such cases. However, constant functions or those with only uniform change are not common in real life application but rather the exception. It is thanks to the work of Isaac Newton and

G.W. Leibniz that realistic situations can be described by the mathematical language they have created, the calculus.

3. The Scientific Revolution

Science in Central Europe had a rather slow start during the Middle Ages. It took Europeans centuries to overcome the language barriers and gain the ability to learn from Arab scholars and through the writings of these from Greek, Egyptian, Mesopotamian, and Indian scientists (Boyer, 1991). Creative work other than just commenting on existing knowledge started during the 12th and 13th centuries. Towards the 17th century Europeans had finally reached a stage where the real potentials could be explored. This period is considered by some the "The Scientific Revolution" for they observe sudden fundamental changes in the way scientists thought about the physical world and the way they studied it (Anonymous, 2009g). Discoveries were now made in quick succession in many different fields of study. W. Harvey and P. Fauchard studied and described the anatomic and physiological properties of the human body, A. van Leeuwenhoek constructed a microscope and became the first person who directly observed bacteria, muscle fibers, and blood flow in capillaries. Astronomers directed their telescopes to the skies and converted their observations into novel theories about our solar system and the universe. J. Kepler presented his three laws of planetary motion and Galileo Galilei observed for the first time some of Jupiter's moons using a telescope he had constructed. Isaac Newton followed in the footsteps of these two and constructed the first practical reflecting telescope which enabled him to make his own astronomical observations. Shortly after, he would formulate his law of universal gravitation and the laws of motion.

As was the case for most other areas of knowledge, mathematics also advanced substantially during the Scientific Revolution. J. Napier introduced the logarithm to simplify calculations,

a transformation of numbers that is an outstanding tool to express any type of numerical scientific data. P. de Fermat developed a method to find extreme points of a curve, discovered the "Fundamental Principle of Analytic Geometry" and formulated several theorems in the area of the theory of numbers. R. Descartes presented his analytic geometry and in a written dialog B. Pascal and P. de Fermat initiated modern theory of probability. G.W. Leibniz described the binary system and he and Isaac Newton both developed independently from each other the calculus.

Science in Central Europe had finally reached a momentum never experienced before and the foundations of almost all of our modern branches of science were built or reinforced during this period. Our two protagonists, Sir Isaac Newton and Gottfried Wilhelm Leibniz participated actively in this exciting period of discoveries.

4. The Protagonists

4.1. **Sir Isaac Newton.** Isaac Newton was born on Christmas day, December 25, of 1642 in Woolsthorpe, England. He grew up as the only child of his widowed mother who gave his son into the care of her own mother when the boy was three years old. Isaac Newton attended public school and was soon known to his classmates as the student who would not join them often in their games, but would rather observe things around him and get engaged in constructing diverse mechanical devices, some invented by himself. Still a student, he constructed for example a fully operational model of a windmill, or a water clock that was in use for years (Chittenden, 1846).

He entered Trinity College at Cambridge at age 18. There he studied mathematics, logic, and optics with texts from Euclid, Descartes, Wallis, Saunderson, Kepler, and others. Early on he made friends with his mentor and professor, Dr. Barrow. Newton received his Bachelor of Arts at age 23 and continued his studies about diverse subjects, including the nature of

light. However, his optical experiments were interrupted by the temporary closure of the College due to an outbreak of the plague. He retired to his hometown Woolsthorpe where he apparently found the adequate environment to meditate and create new ideas. These two years resulted to be extremely fruitful for his studies. He made decisive steps in his major contributions to mathematics and sciences, the binomial theorem, the calculus, the principle of universal gravitation and the nature of colors (Boyer, 1991). Upon his return to Cambridge he resumed his formal studies and received the degree of Master of Arts in 1668 (Chittenden, 1846). A year later he succeeded his mentor and friend Dr. Barrow as Lucasian Professor of Mathematics, an appointment he retained until 1703. In 1695 he accepted a position in the Royal Mint where he took charge of England's recoining. Despite this new occupation he remained actively engaged in research during his whole life. In 1703 he became the President of the Royal Society of London and was reelected to this position each year thereafter until his death. In 1705, the honor of knighthood was conferred to him. In 1699, he became one of the very few foreigners to be elected member of the Royal Academy of Sciences at Paris. This election was most remarkable since, while Newton was very much admired in England, he did not receive the same acceptance on the Continent. Voltaire was quoted with the statement that though Newton survived the publication of his great work more than forty years, yet, at the time of his death, he had not above twenty followers out of England (Chittenden, 1846).

Although already retired and plagued by some physical ailments, at age 85 he undertook the journey to London to attend a meeting of the Royal Society. He died three weeks later on March 20, 1727. The great Sir Isaac Newton is buried at Westminster Abbey in London.

4.2. **Gottfried Wilhelm Leibniz.** Gottfried Wilhelm Leibniz was born on July 1, 1646 in Leipzig, Germany. His father, who was a professor of moral philosophy at the University

of Leipzig, maintained a well equipped private library where his son was granted free access from an early age on (Anonymous, 2009d). Apparently young Leibniz made ample use of this opportunity, for instance by teaching himself Latin which he would use for his writing later in his life. He was admitted to the University of Leipzig at age 14 and spent the following six years studying law and philosophy (Anonymous, 2009e). Upon completion of his studies he was offered a position at the University, but declined. Instead he chose the position of a political adviser for an influential noble German family. The purely theoretical work as a professor didn't really appeal to him, however, he always stayed close to the academy. When in 1672 he was sent to Paris by his employers on a diplomatic mission, he took great advantage of the circumstances and complemented his studies at the University there. It was in Paris where he met Dutch physicist and mathematician Christiaan Huygens (Anonymous, 2009e) who apparently encouraged Leibniz to pursue further studies in mathematics. Leibniz spent four years in Paris and during these years he developed his calculus. As part of his diplomatic activities he travelled to London in 1673. This journey resulted academically eventful for him as he had the opportunity to present to the Royal Society a calculating machine he had been developing for several years. As a consequence of this presentation he was elected a member of this notable organization (Anonymous, 2009d). In 1676, before returning to Germany, he made a second journey to London and it has been speculated that he was granted access to some unpublished works by Isaac Newton during this stay.

His return to Germany brought him a new appointment at the House of the Duke of Brunswick at Hannover. His responsibilities were similar to the ones at his former position and here also he managed to pursue his academic studies alongside his duties as historian, political adviser, and librarian. An extensive correspondency including over 20000 extant letters shows that he maintained an active communication with the scientific community

and he also continued his work on his calculus (Anonymous, 2009d). From the study of his notes it was conjectured that by 1677 he had concluded the latter and he published his results in 1684 (Anonymous, 2009e).

G.W. Leibniz became a member of the Académie des Sciences at Paris in 1669. In 1770 he founded the Berlin Academy of Sciences and became its first president (Anonymous, 2009f). Gottfried Wilhelm Leibniz died in 1716 at the age of 70 in Hannover where he is buried at the Church of Neustädt. The extensive collection of documents, treatises, and letters he had authored during his lifetime is being archived and investigated at the Library of the Federal State of Lower Saxony at Hannover that bears his name. Almost three centuries after the death of this universal and extremely prolific scientist the publication of his written remains is still uncompleted (Anonymous, 2009e).

5. The Different Approaches

Sir Isaac Newton and Gottfried Wilhelm Leibniz made significant contributions to mathematics in general and one of their outstanding discoveries was the calculus. The result of the work of both men's work was similar, yet they started out on very different paths. The reason for the differing approaches can be found in the men's academic backgrounds and hence in their different interests for the subject they were developing.

5.1. **Newton's Dynamic View.** Isaac Newton was a physicist first and a mathematician second. His primary interests were physical phenomena. Among other outstanding contributions to science he formulated the principle of universal gravitation and the laws of motions. However, he was not primarily an astronomer either since it was rather the forces that govern the movement of bodies that intrigued him and not just the distribution of these bodies in the universe. In order to observe his objects of study, the sun, planets, moon and stars, he needed the technology to do so and engaged in producing his own telescopes. It

was during this activity and while studying at Trinity College that he experimented with the reflection and refraction of light which eventually would result in a novel theory about the spectral nature of light (Chittenden, 1846).

When Newton saw lines drawn on paper, he had moving bodies in mind. It is possibly this point of view that enabled him to relate static, abstract features like points, lines, and surfaces with physical objects in motion. A point when in motion creates a line, a line in motion describes a surface, and a surface in motion creates a volume. Very much in the same way as a body in motion traces a path, a spinning piece of rope sweeps over a circle and a circular piece of material when rotated around its diameter spans a body of revolution. Objects in motion flow through space, they are fluents. And while they are moving, their position in space changes over time and the corresponding rate of change was called fluxion by Newton. With this point of view Newton was clearly representing the area of study that Archytas in the 5th century BC had named *magnitudes in motion* or astronomy (Boyer, 1991).

Scientists before Newton had used this physical point of view of mathematical objects. Archimedes of Syracuse lived in Greece almost two millennia earlier and, similar to Newton, he had his mind set primarily on subjects other than mathematics. Archimedes was an engineer and physicist and he experimented with physical media. He then used mathematics to describe what he had found. It is this approach that enabled him to see a line as a physical object and surfaces made up of a collection of lines just like a piece of physical matter could be composed of a collection of sticks joined together next to each other. A physical object hold by its center of gravity is in balance just like a surface made up of lines arranged in a symmetric manner is in balance. Archimedes had called this “The Method” (Boyer,

1991) and, once established, used it extensively to deduce and prove diverse properties of mathematical objects.

Studying the processes related to bodies in motion was also a popular subject in Europe during the Middle Ages. In the early 14th century, mathematicians at Merton College at Oxford compared bodies in uniform motion as well as those undergoing a uniformly accelerated motion and deduced the concept of average speed (Anonymous, 2009a). A short time later, French mathematician Nicole Oresme made the first graphical representation of the speed as a function of time for a motion with constant acceleration (Tashow, 2009) and thereby indirectly related the point in motion as the generator of a line long before Newton used this same idea.

Two centuries later, the Dutch mathematician and engineer, Simon Stevin, studied the equilibrium of mathematical figures very much in the sense Archimedes had done. He used the concept of equilibrium of symmetrically arranged figures to show that the center of gravity of a triangle lies on its medians (Boyer, 1991).

We have here a group of scientists who all were engineers or physicists first but used their vision to give certain mathematical content a new significance. In this sense one could consider Newton, Stevin, Oresme and the scientists at Merton College as mathematicians in the tradition of Archimedes. However, it is Newton who is to be credited with developing the method to perfection and to introduce a consistent notation for it. Oresme had related an equation of a line to the graph of his line, but Newton's notation went far beyond one specific case. He used a letter for what he defined as a fluent, say x, and this fluent can be a position of a body in space but it can, of course, be any other physical parameter. The rate of change of this fluent, i.e. its fluxion, he denoted with a dot above the letter: $\dot{x}$, a notation still in use in physics and engineering for derivatives in terms of time. Oresme's

graph could be described with Newton's notation as: $\ddot{x} = \dot{v} = a =$ constant, where x is the position of the body(i.e. the fluent), v its speed (i.e. the fluxion) and a is the acceleration of the body in motion (i.e. the fluxion of the fluxion).

5.2. **Leibniz' Static View.** When G.W. Leibniz came to Paris in 1672, his academic background was based on philosophy and law. With the universal education he had received it can be conjectured that he had studied some mathematics and physics, but it was certainly not the main focus of his work. Furthermore, there was a considerable difference in quality between German universities and those of other countries, France included. Leibniz had ample experience to self-teach himself new contents from a very young age on and he used these abilities again at Paris. His mentor became Christiaan Huygens (Anonymous, 2009e), a Dutch mathematician and physicist. Huygens by that time had finished his work on his pendulum clock he had invented (Boyer, 1991). The key novelty of this clock was a specific construction that made the pendulum swinging in a cycloidal arc which rendered the motion of the pendulum independent of the height of the mass. Huygens' had studied Apollonius' work on conics extensively and it were probably these concepts that as a mentor he transmitted to Leibniz.

The mathematical approach Leibniz became familiar with during his time in Paris was the geometric viewpoint. Even if problems with moving bodies were treated, the positions and motions of these bodies were sketched in curves as if one would take a snapshot of the motion and study this picture. This viewpoint coincides best with the area that Archytas in the 5th century BC defined as *magnitudes at rest* or geometry, and which Apollonius studied extensively in his theoretical work about conics two centuries later.

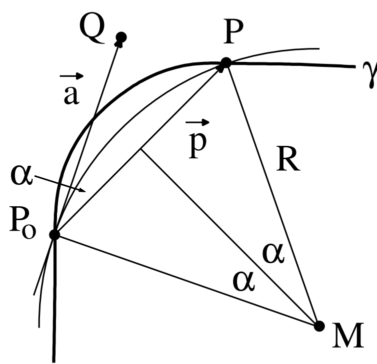
The problem of finding the tangent to a curve caused much attention among mathematicians during the 17th century and Leibniz followed Pierre de Fermat, Blaise Pascal and Isaac

Barrow in their respective studies. All of these men used a similar static concept of the tangent line as studied by Apollonius (Boyer, 1991). They all traced the tangent line to a specific point of a curve as part of a geometric construction, then drew a right triangle in such a way that the hypothenuse of it coincided with the tangent line. They then studied the geometry of this right triangle as related to the curve and the coordinate axes. The principal tools for this type of treatment are taken from trigonometry and mainly involve the comparison of similar triangles found in the respective geometric setup. As compared to the dynamic approach that Newton had used, Leibniz approach was not based on any other motivation than on pure mathematics.

5.3. **Common Concepts.** Indifferently from the motivation Newton and Leibniz had for their development of the calculus, they both used the same mathematical tools. The principal argument that both based their reasoning on were the use of infinite power series. They both considered very small quantities for the variation of the variables and disregarded higher order of these when they appeared in the terms of these series. Newton had extended the binomial theorem to rational exponents as one of his first steps to develop his calculus. Since he communicated this result to the Royal Society in 1676, it was also available to Leibniz at least during the later stage of his work on the calculus. The main achievement of their combined work is the generalization of the procedure of finding tangents and integrating curves, procedures that apply equally to all different types of functions including transcendental functions.

The determination of the curvature of a curve using first the geometric setup and then the dynamic concept will serve as an example to illustrate the two approaches.

We want to obtain the curvature at the point P_o that lies on a smooth curve γ. Let P be some other point on γ. A circle is constructed through P_o and P with center at M and radius R as shown in the following figure:



If we locate the center of our coordinate system in the point of interest, P_o, then $\vec{p}$ indicates the location of P. Furthermore, let P_oQ be the tangent to the circle at point P_o and $\vec{a}$ the location of some point Q on the tangent. Let γ be given in parametric form, where $\gamma(t = 0) = P_o$ and $\gamma(t) = P$. The curvature at P_o is then defined by the limit

$$\kappa = \lim_{t \to 0} \frac{1}{R(t)} = \frac{1}{R_o}$$

In order to find this limit, we use a geometric approach. Greek geometers have shown that the angle of the arc P_oP is twice the angle formed by the tangent at P_o and the chord P_oP. Hence

$$\cos\alpha = \frac{\vec{a} \cdot \vec{p}}{|\vec{a}||\vec{p}|} \quad \Rightarrow \quad \cos^2\alpha = \frac{(\vec{a} \cdot \vec{p})^2}{|\vec{a}|^2 |\vec{p}|^2}$$

In the triangle $\triangle P_oMP$ we have:

$$R \sin\alpha = \frac{|\vec{p}|}{2} \quad \Rightarrow \quad \sin^2\alpha = \frac{|\vec{p}|^2}{4R^2}$$

By using the identity $\sin^2\alpha + \cos^2\alpha = 1$ we can solve these for R by eliminating α:

$$\cos^2\alpha = 1 - \sin^2\alpha \quad \Rightarrow \quad \frac{(\vec{a}\cdot\vec{p})^2}{|\vec{a}|^2|\vec{p}|^2} = 1 - \frac{|\vec{p}|^2}{4R^2} \quad \Rightarrow \quad \frac{1}{R^2} = \frac{4}{|\vec{a}|^2}\cdot\frac{|\vec{a}|^2|\vec{p}|^2 - (\vec{a}\cdot\vec{p})^2}{|\vec{p}|^2|\vec{p}|^2} \quad (*)$$

We now assume that the curve $\gamma(t)$ is analytic, i.e. it can be expressed as the polynomial:

$$\vec{p} = \gamma(t) = \vec{a}\cdot t + \vec{b}\cdot t^2 + \vec{c}\cdot t^3 + \cdots$$

We have:

$$\gamma'(t) = \vec{a} + 2\vec{b}\cdot t + 3\vec{c}\cdot t + \cdots \quad \text{where} \quad \gamma'(0) = \vec{a}$$

$$\gamma''(t) = 2\vec{b} + 6\vec{c}\cdot t + \cdots \quad \text{where} \quad \gamma''(0) = 2\vec{b}$$

We can now compute all the terms needed for our equation $(*)$ above by simply multiplying out polynomials. We thus obtain for example for $(\vec{a}\cdot\vec{p})^2$:

$$\begin{aligned}(\vec{a}\cdot\vec{p})^2 &= \big(\vec{a}(\vec{a}t + \vec{b}t^2 + \vec{c}t^3 + \cdots)\big)^2 \\ &= (|\vec{a}|^2 t + (\vec{a}\cdot\vec{b})t^2 + (\vec{a}\cdot\vec{c})t^3 + \cdots)^2 \\ &= |\vec{a}|^4 t^2 + 2|\vec{a}|^2(\vec{a}\cdot\vec{b})t^3 + (2|\vec{a}|^2(\vec{a}\cdot\vec{c}) + \big(\vec{a}\cdot\vec{b})^2\big)t^4 + \cdots\end{aligned}$$

In the same manner:

$$|\vec{p}|^2|\vec{p}|^2 = |\vec{a}|^4 t^4 + 4|\vec{a}|^2\big(\vec{a}\cdot\vec{b}\big)t^5 + \cdots$$

$$|\vec{a}|^2|\vec{p}|^2 = |\vec{a}|^4 t^2 + 2|\vec{a}|^2\big(\vec{a}\cdot\vec{b}\big)t^3 + \big(2|\vec{a}|^2(\vec{a}\cdot\vec{c}) + |\vec{a}|^2|\vec{b}|^2\big)t^4 + \cdots$$

If we substitute the terms in $(*)$ by these infinite power series and simplify we obtain:

$$\frac{1}{R^2} = \frac{4}{|\vec{a}|^2}\cdot\frac{\big(|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2\big)t^4 + (\cdots)t^5 + \cdots}{|\vec{a}|^4 t^4 + 4|\vec{a}|^2(\vec{a}\cdot\vec{b})t^5 + \cdots}$$

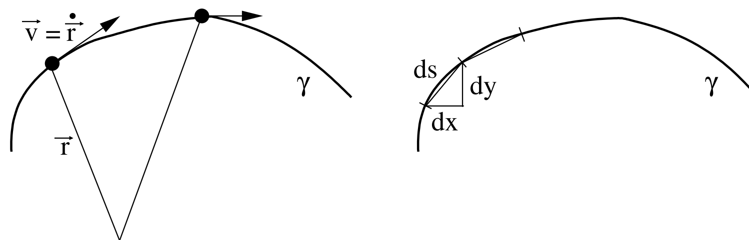
After dividing through by t^4 we can now compute the limit to obtain:

$$\begin{aligned}\kappa = \lim_{t\to 0}\frac{1}{R(t)} &= \lim_{t\to 0}\left\{\frac{4}{|\vec{a}|^2}\,\frac{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 + (\cdots)t + \cdots}{|\vec{a}|^4 + 4|\vec{a}|^2(\vec{a}\cdot\vec{b})t + \cdots}\right\}^{1/2} \\ &= \left\{\frac{4(|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2)}{|\vec{a}|^6}\right\}^{1/2}\end{aligned}$$

We remember that $\gamma'(0) = \vec{a}$ and $\gamma''(0) = 2\vec{b}$ and finally obtain for the curvature at point P_o:

$$\kappa = \frac{1}{R_o} = \frac{\sqrt{|\gamma'|^2|\gamma''|^2 - (\gamma' \cdot \gamma'')^2}}{|\gamma'|^3}$$

We can find the same formula for the curvature by using a physical argument as follows. A mass point moves on a smooth path γ; let its position at a certain time t be given by $\vec{r} = (x(t), y(t), z(t))$. The curvature is defined by $\kappa = |\gamma''(s)|$ where the parameter s is the distance the mass point has travelled on our path γ.



Obviously, s also depends on the time t. The curvature is then the second derivative of the position $\vec{r}$ in terms of the distance the mass point has travelled on the path:

$$\kappa = \left|\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}s^2}\right|$$

We make use of the chain rule and obtain for this second derivative:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{\mathrm{d}\vec{r}}{\mathrm{d}s}\Big) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\Big) = \Big(\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}\frac{\mathrm{d}t}{\mathrm{d}s}\Big)\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{\mathrm{d}t}{\mathrm{d}s}\Big) \\ &= \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}\Big(\frac{\mathrm{d}t}{\mathrm{d}s}\Big)^2 + \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathrm{d}t}{\mathrm{d}s}\Big) \qquad (**)\end{aligned}$$

We can identify all factors of the above identity in the context of the physical model of the mass point in motion as follows:

$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \dot{\vec{r}} = \vec{v}$: the velocity of the mass point along its path

$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t} = \ddot{\vec{r}}$: the derivative of the velocity v, i. e. the acceleration in the direction of the motion

$\frac{\mathrm{d}s}{\mathrm{d}t} = |\vec{v}|$: the distance $\mathrm{d}s$ on the path the mass point can move in a small amount of time $\mathrm{d}t$, i.e. the absolute value of its velocity

This absolute value of the velocity is $|\vec{v}| = |\dot{\vec{r}}|$, hence

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\dot{\vec{r}}| = \sqrt{\dot{\vec{r}}^2} \qquad \text{and} \qquad \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \dot{\vec{r}}^2$$

Then also:

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-1} = \left(\sqrt{\dot{\vec{r}}^2}\right)^{-1} \qquad \text{and} \qquad \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = (\dot{\vec{r}})^{-2}$$

We replace all this in (∗∗) and obtain:

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}s^2} = \ddot{\vec{r}} \cdot \frac{1}{\dot{\vec{r}}^2} + \dot{\vec{r}} \frac{1}{\sqrt{\dot{\vec{r}}^2}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{\dot{\vec{r}}^2}}\right)$$

We perform the derivative in the second term using the chain rule and obtain:

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}s^2} = \frac{\ddot{\vec{r}}}{\dot{\vec{r}}^2} - \frac{\dot{\vec{r}}}{\dot{\vec{r}}^4}\left(\dot{\vec{r}} \cdot \ddot{\vec{r}}\right)$$

For the curvature, we need the absolute value of this, so take the square:

$$\left(\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}s^2}\right)^2 = \frac{\ddot{\vec{r}}^2}{\dot{\vec{r}}^4} - 2\frac{(\dot{\vec{r}} \cdot \ddot{\vec{r}})^2}{\dot{\vec{r}}^6} + \frac{\dot{\vec{r}}^2(\dot{\vec{r}} \cdot \ddot{\vec{r}})^2}{\dot{\vec{r}}^8} = \frac{\dot{\vec{r}}^2\ddot{\vec{r}}^2 - (\dot{\vec{r}} \cdot \ddot{\vec{r}})^2}{\dot{\vec{r}}^6}$$

We finally obtain for the curvature:

$$\kappa = \sqrt{\left(\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}s^2}\right)^2} = \frac{\sqrt{|\dot{\vec{r}}|^2|\ddot{\vec{r}}|^2 - (\dot{\vec{r}} \cdot \ddot{\vec{r}})^2}}{\dot{\vec{r}}^3}$$

which is identical to the result we obtained using the geometric approach if we remember that $\gamma(t) = \vec{r}(t)$.

6. The Calculus Controversy

Apparently the time was ripe for the development of the calculus and two extraordinary scientists coincided in time but at different places and undertook the next steps in what we now know were virtually two almost independent processes (Anonymous, 2009d).

Newton developed his calculus during his two very productive years 1665/1666 at his hometown Woolsthorpe where he had to retire while Trinity College at Cambridge was closed due to the Great Plague. By the same time Leibniz in Germany was finishing his studies in philosophy and law. Upon his return to Cambridge, Newton pursued his studies on optics, gravitation, and mathematics, but quickly assumed full academic and administrative responsibilities related to his new position as the Lucasian Professor of Mathematics at Trinity College. The same year, 1669, Leibniz was appointed Assessor in a Court of Appeal and was concerned with legal codes rather than mathematics. On the other hand, Newton communicated for the first time his results related to the calculus in a private letter to his mentor and friend, Dr. Barrow, who shared the document with the then librarian of the Royal Society, John Collins (Chittenden, 1846). It is well known that Mr. Collins corresponded extensively with many European scientists (i.e. O'Connor and Robertson, 2009) including the leading mathematicians of the time, David Gregory, Christiaan Huygens, and John Wallis, and eventually also G.W. Leibniz. When Leibniz visited London for the first time in 1973, he met with Mr. Collins and during his second visit three years later, Mr. Collins apparently granted Leibniz access to some unpublished documents authored by Isaac Newton. By this time, however, Leibniz had already made essential advances in the development of his own calculus and as exposed above, his approach varied substantially from that of Newton. It is not confirmed by any of the cited documents herein that the possible (not even proven!) access to Newton's unpublished works had any significant influence on

Leibniz' own work on the calculus. Leibniz returned to Germany where he had to focus on his new appointment at Hannover. Several years later, namely in 1684, appeared in print his first public account about his calculus in the "Acta Eruditorium", a scientific journal that had been founded two years earlier in Leipzig. It took another three years until Newton published his "Philosophiae naturalis principia mathematica" where he included his first limited account of his calculus (Newton, 1846). A comprehensive description was published much later in 1693 by John Wallis (Chittenden, 1846).

With the publication of Leibniz' results in the "Acta Eruditorium" in 1684 and without any information available publicly from Newton, the scientific community on the Continent apparently came to believe that Leibniz was the exclusive author of the new calculus. In 1695, John Wallis communicated this observation to Isaac Newton and it was soon the general belief in England that Leibniz had to have committed plagiarism by borrowing ideas from the documents Leibniz prosumibly had the opportunity to review during his second journey to London. Of course, rumors in this sense became known on the Continent and the controversy went into its critical stage. In 1704 in a letter to the Royal Society (whose president was at that time Isaac Newton), Leibniz protested against the imputations which prompted the Society to create a committee to investigate in the matter. It was only in 1713 that this committee reported to the scientific community the results of this investigation. The members had come to the conclusion that Newton was actually the first inventor of the new method, a fact that Leibniz himself had never negated. His protest concentrated on the accusation of plagiarism which the report could neither definitely confirm nor deny, but only state the possibility due to the chronology of the events (Chittenden, 1846). When Leibniz died three years later the controversy had not been resolved and it even persisted

between the English and Continental scientific communities long after Newton's death in 1727 (Anonymous, 2009d).

It is a somber fact that the controversy between Newton and Leibniz and their respective supporters about the priority of merit for the development of the calculus came to overshadow the work of these two brilliant minds. Varying reasons for this controversy have been presented over time. It appears that back then the procedures to publish scientific results were not standardized as they are nowadays. When Newton and Leibniz were ready to present their results, most of the communication within the scientific communities still occurred by means of private letters and the publication in journals and as books were rather isolated events. Only later were scientific journals founded like the "Acta Eruditorium" that Leibniz used.

Considering the chronology of biographic events it is furthermore clear that both men upon finishing their respective calculus were mainly focusing on activities other than calculus. Although Newton's main occupation was academic in nature, his mind was set on the development of his theory of universal gravitation. He might not have seen any urgent need to publish the calculus independently since he had developed it as a tool for his main studies. Leibniz' situation was different than Newton's. When he concluded most of his calculus in 1676, his time in Paris had come to an end and he had to return to Germany. There he had to focus on a new, challenging and time consuming appointment which might have delayed the publication of his results. However, once the "Acta Eruditorium" had been founded as the first scientific journal in Germany, he relatively quickly used this novel, convenient way of communicating results and became a frequent contributor to the journal.

These circumstances might explain to some extend that although Newton had concluded the work on the calculus much earlier than Leibniz, the latter anticipated the former in

its publication. The ongoing generalized rivalry between the English and the Continental scientists seemed to have led these to take advantage of the event to stage yet another controversy.

7. There is much more than calculus!

The calculus that Isaac Newton and G.W. Leibniz developed has had in immense impact on mathematics and science ever since. Their work pushed the door wide open to explore functions and fields of all types and dimensions including general, non-uniform behavior. Both, Newton and Leibniz, contributed much more than the calculus, though. Isaac Newton's law of universal gravitation and the laws of motion set the stage for modern astronomy, physics, and engineering. His theories were so grandiose and innumerable experiments provided measurements that confirmed these laws so consistently that it was unthinkable that they would ever be questioned. It took scientists over two centuries to realize that Newton's theories of gravitation were actually only a special case of a yet more general theory, namely Albert Einstein's general theory of relativity. Science has made a full circle in that Newton's actual successor at the Lucasian Professorship at the University of Cambridge is Stephen W. Hawking, a theoretical physicist and applied mathematician like Newton had been himself. G.W. Leibniz was a truely universal thinker who did not settle in any specific subject area. Besides mathematics, his contributions are found in areas as diverse as geology, philosophy, law, engineering, physics, history, and linguistic. His presence in the collective consciousness has not the same strength as Newton's, however this does not decrease the impact of his work in the least. His study and presentation of the binary system together with his work on logic represent the foundation of the computer sciences which provide the framework for virtually any aspect of our present life.

Newton's discovery of the nature of color and the already mentioned binary system that Leibniz presented are only two examples of a long list of contributions by these two scientists.

7.1. **The Nature of Color.** Newton's discovery of the nature of color originated in his failure to grind optic glasses of other figures than spherical ones in his attempt to improve refracting telescopes. He was obviously very much convinced of his faculty of grinding lenses since he attributed the poor results not to his own lack of manual dexterity but to some physical phenomena he had not yet taken into account. He conducted experiments with a triangular glass prism and found his assumptions confirmed. The results led him to the conclusion that white light was composed of different rays that were refracted at different angles when passing the air-glass limit. He went one step further and concluded that there were no means to produce a telescope lens that would focus all light rays coming from a single source into a single point, hence the project of producing refracting telescopes had to be abandoned altogether. More experiments showed him that the direction to proceed was to use the principle of light reflection (Chittenden, 1846). Unlike refraction, in the case of reflection the angle was not dependent on the specific color of light. His studies on optics where interrupted by a two years retreat to Woolsthorpe while the College was closed due to an outbreak of the plague in England, but in the year of 1668 he returned to Cambridge and resumed his optical experiments. He took on the task and soon succeeded in constructing the first reflection telescope that was directed to the skies (Chittenden, 1846). In 1704 he published a comprehensive work on his studies about optics under the title of "Optics: or a Treatise on the Reflections, Refractions, Inflexions and Colours of Light".

The way this fundamental discovery was made is most typical for Newton's style. He apparently had set his mind on a certain goal, in this case the observations of the skies that would eventually provide him with information he needed for his research on gravitation.

With the tools available to him at this time, his goal was unreachable, hence he produced all necessary tools himself in order to proceed. The discovery of the nature of color was as much a side-product of his studies towards his higher goal to explain the system of the world as was the development of the mathematical tool he needed, the calculus. Needless to say that these side-products reoriented the science of optics as well as of mathematics.

7.2. **Binary System.** Many different number systems have been and are still in use over the millennia (Boyer, 1991) including the base-2 (binary) system. Yet Leibniz was the first to fully describe and explain the latter system in a work entitled "Explication de l'Arithmétique Binaire" published in 1703. Leibniz had expressed in several occasions his belief that time spent on simple calculations was an unworthy activity for any individual who was pursuing higher goals, hence the idea to construct a machine that would free men from this unfortunate state of slavery (Liebezeit, 2009).

In 1673 he presented his calculating machine, the "Step Reckoner" to the Royal Society at London which earned him the membership in this honorable society. This machine operated based on the decimal system and could perform all four basic arithmetic operations. However, during the following years he studied the binary system and in several occasions in his writings he described the possible design for alternative calculating machines based on the binary system (Anonymous, 2009e). Although he never realized the construction of any of these, he nevertheless had created the conceptual framework for the nascent discipline of computer sciences. It is the characteristics of the binary system that all operations can be reduced to simple yes/no decisions. While this fact favors the construction of mechanical machines, it is specifically advantageous for the modern digital calculators, since it proved to be easy to find simple electronic devices that perform the required basic yes/no decisions.

8. Conclusions

The calculus as developed by Isaac Newton and Gottfried Wilhelm Leibniz has had an enormous impact on mathematics and science. There are two approaches to the subject that ultimately lead to identical general results, the dynamic approach that Newton has developed and the static, geometric approach used by Leibniz. The former finds its applications in the fields of physics and engineering, while the latter is used mainly in pure mathematics. The coexistence of both approaches offer the possibility to choose the more convenient one for each specific application and it has been my observation in science as well as in mathematics that people take advantage of this possibility of choice and frequently switch between both.

The calculus is by far the most important branch of mathematics for most areas in science and engineering. Any research or application in either of these areas depends to a high degree on the use of the calculus. The predominance of this part of mathematics is such that calculus is perceived by many scientists and more so by engineers as mathematics per se. With my own background in science it has been an exciting and extremely enlightening process to dig into the origins and the development of the calculus and to learn about the two protagonists who created it. While the physicist Newton is omnipresent in almost all aspects of science, little is known among scientists about his being a mathematician. The name of Leibniz is well known in Central European societies, but few know about the actual extent of his contributions to science and mathematics and if at all most would relate him with philosophy rather than mathematics. This work has helped me to set things straight a bit and left behind the impression that there is so much more to explore.

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So many beautiful quotes from Newton! I especially liked these two:

"I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, while the great ocean of truth lay all undiscovered before me."

" Let ... it being the design of travellers to learn, not to teach."

— Sir Isaac Newton —