A Complex H*i*story

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Overview

- We will explore the evolution of the number *i*. The discovery of the square root of negative one had repercussions on many aspects of mathematics. We will examine some of these effects.

Some definitions:

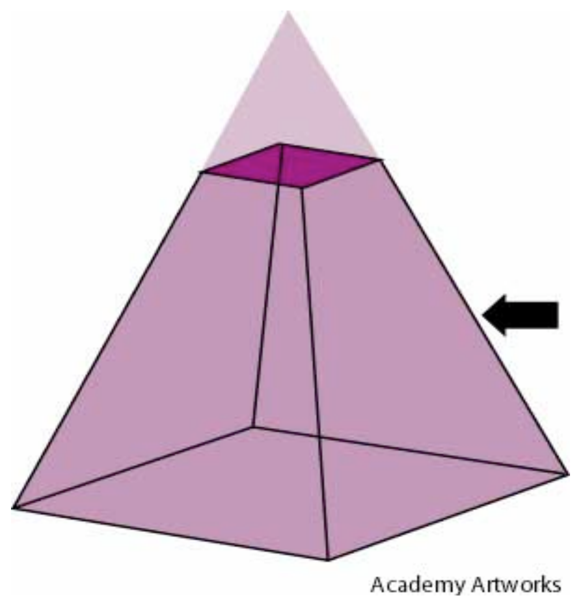
- A <u>complex number</u> n can be defined as:

  $n = x + iy$, where x and y are real numbers and i is the imaginary unit.
- An <u>imaginary number</u> is a complex number whose real part is equal to zero.
- i can be defined by the property: $i^2 = -1$

A difficult beginning…

- The earliest reference to a square root of a negative number is thought to have been in the 1^{st} century AD by Heron of Alexandria.

- He happened upon it when he was trying to find the volume of a frustum of a pyramid with a square base.
  - A frustum is a portion of a solid that lies between two parallel planes



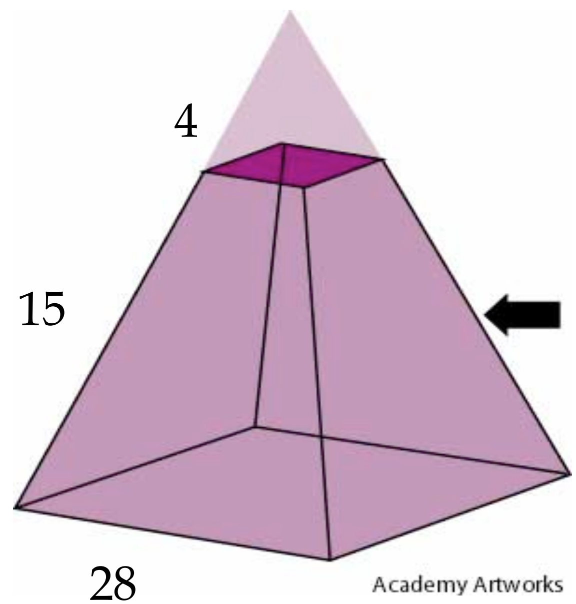
Heron:

- Here's what he did:
  - $$h = \sqrt{c^2 - 2\left(\frac{a-b}{2}\right)^2}$$

$$= \sqrt{(15)^2 - 2\left(\frac{28-4}{2}\right)^2}$$

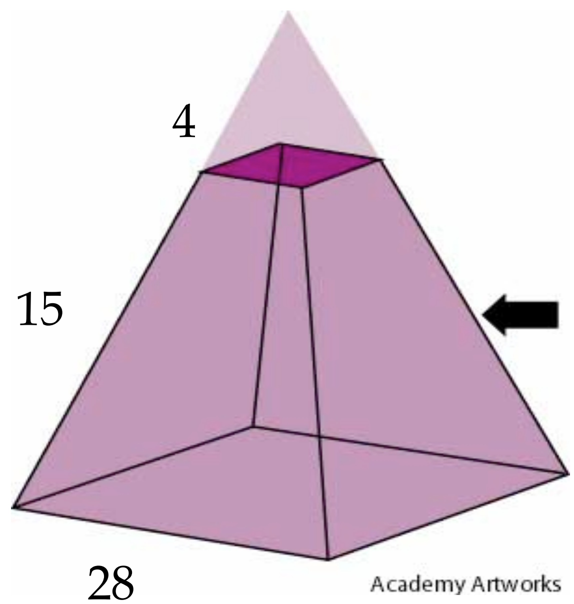
$$= \sqrt{225 - 288}$$

$$= \sqrt{-63}$$



Heron:

- Rather than taking the square root of -63, Heron disregarded the negative and took the square root of 63 instead.



- Nearly two centuries later, Diophantus of Alexandria essentially did the same thing.
  - When his quadratic equation gave him the square root of -167 as part of his solution, he disregarded it as impossible.
- This pattern continued.
  - Negative numbers themselves were oftentimes rejected

Cardano and Tartaglia

- Imaginary numbers were discovered in the 16^{th} century by Gerolamo Cardano (1501-1576) and Niccolo Fontana Tartaglia (1499-1557).

- They discovered imaginary numbers with closed formulas for roots of cubic equations.
- Imaginary numbers, however, were not widely accepted until the math of Leonhard Euler (1707-1783) and Carl Friedrich Gauss (1777-1855).

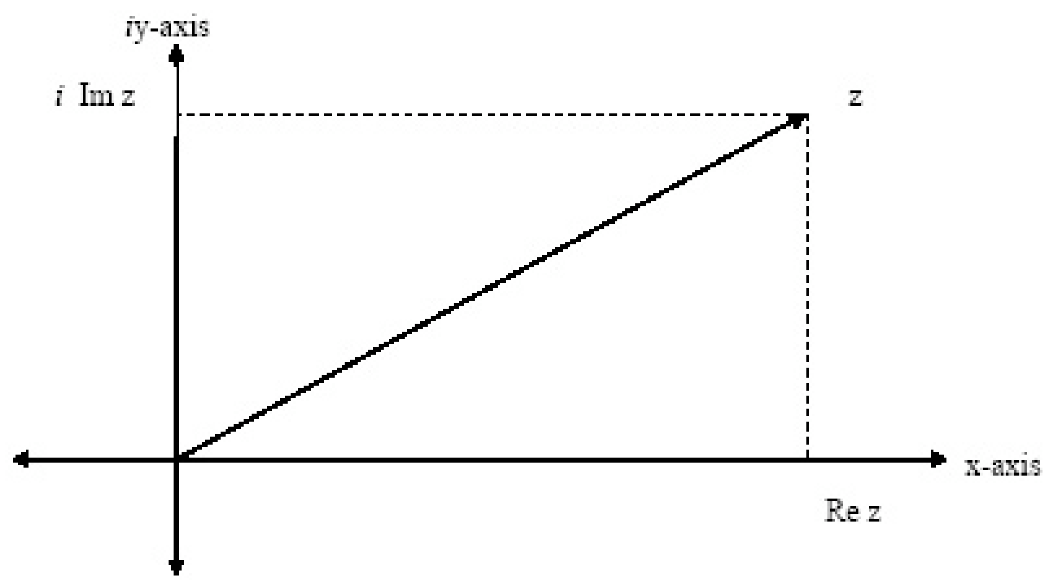
Some more history

- Rafael Bombelli (1526-1573) made great strides for imaginary numbers.
  - Manipulate with algebra.
  - Defined $i^2 = -1$.

- The term “imaginary” was first used by René Descartes (1596-1650) in his *La Géométrie*.

Geometric Interpretation:

- There exists a line perpendicular to the real number line at zero, which increases positively in the upward direction and negatively in the downward direction.



Geometrically

- Introduced in 1799 by mathematician Caspar Wessel (1745-1818).
  - Published, but pretty much unnoticed.
- Abbé Buée
  - came up with the same idea separately in 1804
  - published it in 1806.
- That same year, Jean-Robert Argand also geometrically interpreted imaginary numbers.

- It was later rediscovered, and popularized, by Carl Friedrich Gauss in 1831.

*i*nteresting Equations

$$e^{ix} = \cos x + i \sin x$$

$$i^i = e^{-\pi/2}$$

Proof of $e^{ix} = \cos x + i \sin x$

Define function f by : $f(x) = \dfrac{\cos x + i \sin x}{e^{ix}}$

$$f'(x) = \frac{(-\sin x + i \cos x)e^{ix} - (\cos x + i \sin x)ie^{ix}}{\left(e^{ix}\right)^2}$$

$$= \frac{-\sin xe^{ix} + i \cos xe^{ix} - i \cos xe^{ix} - i^2 \sin xe^{ix}}{e^{2ix}}$$

$$= \frac{-\sin x - i^2 \sin x}{e^{ix}}$$

$$= \frac{-\sin x + \sin x}{e^{ix}}$$

$$= 0.$$

Thus, $f(x)$ is a constant function.

Thus, $f(x) = f(0) = \dfrac{\cos(0) + i\sin(0)}{e^{0}} = 1$

$$\frac{\cos x + i\sin x}{e^{ix}} = 1$$

$$\cos x + i\sin x = e^{ix}$$

□

$e^{ix} = \cos x + i \sin x$

- Provides alternate definitions:
  - $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$
  - $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

$e^{ix} = \cos x + i \sin x$

- A special case, where $x = \pi$
  - This is the famous Euler identity
    - $$e^{i\pi} + 1 = 0$$

Proof of $i^i = e^{\frac{-\pi}{2}}$

Consider a circle of radius 1 centered at the origin :

$$x^2 + y^2 = 1$$

Calculate the area of the 1st quadrant :

$$-\text{which we know to be } \frac{\pi}{4}$$

$$\text{Area} = \int_0^1 y\,dy = \int_0^1 \sqrt{1 - x^2}\,dx$$

Substitute $x = -iu$, and so $dx = -i\,du$

$$\text{Area} = \int_0^i \sqrt{1-(-iu)^2}\,(-i\,du)$$

$$= -i\int_0^i \sqrt{1+u^2}\,du$$

Using the integral tables, we know :

$$\text{Area} = \int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + C$$

$$\text{Area} = \frac{\pi}{4} = -i\left(\frac{1}{2}u\sqrt{u^2+1} + \frac{1}{2}\ln\left(u + \sqrt{u^2+1}\right)\right)\Bigg|_0^i$$

$$= -\frac{1}{2}i\ln(i)$$

$$\frac{\pi}{4} = -\frac{1}{2}i\ln(i)$$

$$\frac{\pi}{4} = \frac{-1}{2} i \ln(i) = \ln\left(i^{i\left(\frac{-1}{2}\right)} \right)$$

$$e^{\frac{\pi}{4}} = i^{i\left(\frac{-1}{2}\right)}$$

$$e^{\frac{-\pi}{2}} = i^{i}$$

□

*i*nteresting facts:

- Zero is the only number that is both real and imaginary.
- In electrical engineering, the imaginary unit is j rather than i.

- “Complex Number Computer”
  - Created by Stibitz between 1938-1940
  - One of the first computers
  - Designed specifically to multiply and divide complex numbers
    - Network analysis
    - Telephone switching problems

More *i*nteresting facts:

- There was confusion in the fact that the equation

$$\sqrt{-1}^{2} = \sqrt{-1}\sqrt{-1} = -1$$

was inconsistent with the algebraic identity

$$\sqrt{a}\sqrt{b} = \sqrt{ab}$$

Euler's Blunder

- Euler actually used this identity incorrectly and argued $\sqrt{-1}\sqrt{-4} = \sqrt{4} = 2$
  - when in fact $\sqrt{-1}\sqrt{-4} = i\sqrt{1}i\sqrt{4} = i^2 2 = -2$

i in Math and Sc*i*ence

- Electrical Engineering
- Quantum Mechanics
- Calculus
- Physics
- Fractals
- Applied math
- Control theory

Bibliography

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