A Complex H$i$story

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Abstract

We will explore the evolution of the number i. the discovery of the square root of negative one had repercussions on many aspects of mathematics. We will examine some of these effects.

1 Introduction

We will discuss the earliest references to square roots of negative numbers and how they were ignored until the mathematics of Cardano and Tartaglia in the 16th century. We will delve into the importance of a geometric interpretation of imaginary numbers and how it effected the final acceptance of the numbers. A complex number n can be defined to be $n = x + iy$, where x and y are real numbers and i is the imaginary unit. An imaginary number is a complex number whose real part is equal to zero. i itself can be defined by the property: $i^2 = -1$.

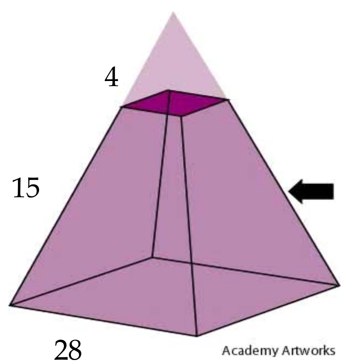
2 A Difficult Beginning

2.1 Heron

The earliest reference to the square root of negative numbers was Heron of Alexandria in the 1st century AD. He came upon the square root of a negative number while trying to find the volume of the frustum of a pyramid with a square base of a certain size. A frustum is a solid between two parallel planes, thus effectively cutting the solid. The frustum of a pyramid looks like a pyramid without a top (see Figure 1). Heron tried to find the area of a frustum of a pyramid with a base length of 28, a side length of 15, and a top edge length of 4. Using this formula

$$h = \sqrt{c^2 - 2\left(\frac{a-b}{2}\right)^2},$$

Figure 1: Frustum of a Pyramid



Heron derived the answer to be the square root of -63 by computing the following:

$$h = \sqrt{c^2 - 2\left(\frac{a-b}{2}\right)^2}$$

$$= \sqrt{15^2 - 2\left(\frac{28-4}{2}\right)^2}$$

$$= \sqrt{225 - 288}$$

$$\sqrt{-63}$$

Instead of calculating a negative square root, Heron merely ignored the negative sign and took the square root of 63.

2.2 The Pattern continues

Nearly two centuries after Heron, Diophantus of Alexandria had the same problem with square roots of negative numbers. While trying to find the solution to the equation

$$172x = 336x^2 + 24,$$

Diophantus found the root to be

$$x = \frac{43 + \sqrt{-167}}{168}.$$

Unable or unwilling to comprehend the meaning of the square root of a negative number, he simply stated the quadratic equation was impossible. The evolution of imaginary numbers was further hindered by the fact that negative numbers themselves were oftentimes rejected due to mathematicians being unable to physically interpret numbers that were less than nothing.

3 A Mathematical Discovery

3.1 Cardano and Tartaglia

The story of the square roots of negative numbers continues in the 16th century with Scipione Del Ferro(1465-1526) and his solution to depressed cubics. Unlike the general cubic equation, such as $x^3 + zx^2 + px = q$, depressed cubics have the second degree term missing. The general form of depressed cubics is $x^3 + px = q$, where both p and q are non-negative, because mathematicians in the 16th century continued to have an aversion to negative numbers.

Like mathematicians in the past, Del Ferro was content with finding a single positive root of an equation, not bothering with the posibility of multiple or negative solutions. Del Ferro successfully kept this method a secret from all but a few close friends up until his death in 1526. One such friend was Antonio Maria Fior, a student of his and not a particularly good mathematician. Fior saw this knowledge as a great opportunity for fame, and preceded to challenge Tartaglia (ca. 1499-1557), or Nicolo Fontana of Brescia as he was also known, to a public contest of problem solving in 1535.

Now, while Del Ferro's method was unknown, the fact that he know how to solve depressed cubic equations was common knowledge among the mathematics world. After being challenged by Fior, Tartaglia correctly assumed that Fior had received this secret information from his former teacher before his death. The prospect of public humiliation and knowing that he was unable to solve the depressed cubic equations prompted Tartaglia to rediscover the solutions himself, a feat in which he was successful. With his discovery, Tartaglia easily defeated Fior.

Much like Del Ferro, Tartaglia fully intended to keep his solution a secret, intent on publishing it in the future (a goal which he never completed). News of Tartaglia's victory over Fior spread, eventually spiking the interest of Girolamo Cardano(1501-1576), also known as Cardan. Cardan's curiosity about the method prompted him to beg Tartaglia to reveal his secret. It wasn't until giving a vow of secrecy, Tartaglia yielded to the insistance of Cardan, giving him the rule, but not the derivation.

Cardan had every intention of keeping his promise to Tartaglia, until Cardan found out that Tartaglia was not the first to solve the depressed cubic. He discovered that Del Ferro had been the mathematician to originally discovered how to solve for solutions of depressed cubic equations. Upon seeing Del Ferro's papers, Cardan no longer felt obligated to Tartaglia to keep the solution a secret. After rediscovering Tartaglia's solution for himself, Cardan proceeded to extend it to include not merely depressed cubics, but all cubics, an achievment all his own. In 1545, Cardan published his new findings in his book *Ars Magna*, being sure to give both Del Ferro and Tartaglia specific credit. This of course did not appease Tartaglia, who resented Cardan for breaking his vow.

3.2 Formalities

Italian mathematician, Rafael Bombelli, made great strides in the understanding of imaginary and complex numbers. Bombelli comtinued the work of Cardan and successfully used algebra to manipulate imaginary numbers. In his algebra, Bombelli also introduced $-i$ and $+i$, and even defined $i^2 = -1$. Though Bombelli gave a formal definition to $\sqrt{-1}$ and utilized successful mathematics with square roots of negative numbers, the idea was still not widely accepted. René Descartes, a French mathematician born into lower nobility, went so far as to call the numbers "imaginary" in his book *La Geometre*. He meant the term to be derogatory, implying that they were fake numbers. A geometrist by trade, Descartes concluded that because there was no geometric interpretation of imaginary numbers, the numbers do not exist.

3.3 Hamilton

Sir William Rowan Hamilton (1805-1865), an Irish mathematician from Dublin, discovered, in 1843, an algebra in which the commutative law of multiplication does not hold, quaternions. Hamilton considered that fact that because a complex number $x + iy$ is determined by the real numbers x and y, complex numbers could be written as ordered pairs. With this new idea, Hamilton attempted to create a number system for vectors and rotation in 3-space. The complex number system proved to be useful in this endeavor.

Rather than using ordered pairs of real numbers, like originally planned, Hamilton thought of utilizing "ordered real number quadruples (a, b, c, d) having both the real and the complex numbers embedded within them." [2] It is these urdered quadruples that are known as real quaternions.

Due to the real and complex numbers being embedded in them, the definition of addition and multiplication of quaternions are as follows:

$$(a, b, c, d) + (e, f, g, h) = (a + e, b + f, c + g, d + h),$$

and

$$(a, b, c, d)(e, f, g, h) = (ae - bf - cg - dh, af + be + ch - dg, ag + ce + df - bh, ah + bg + de - cf).$$

Similar to ordered pairs, quaternions are additively commutative and associative. They are also distributive and associative over multiplication. They are not, however, commutative over multiplication, the very reason for Hamilton's idea. For instance, with the two quaternions $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$, one finds that

$$(0, 1, 0, 0)(0, 0, 1, 0) = (0, 0, 0, 1)$$

while

$$(0, 0, 1, 0)(0, 1, 0, 0) = (0, 0, 0, -1) = -(0, 0, 0, 1),$$

and thus, quaternions are not commutative over multiplication.

Actually, if we represent the quaternionic units $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$ with the symbols $1, i, j$, and k, we obtain the following multiplication table (see Figure 2).

In the table, "the desired product is found in the box common to the row headed by the first factor and the column headed by the second factor." [2] It was found that by using these quaternionic multiplication tables, a plethora of varying algebras could be created.

Figure 2: Quaternionic Multiplication Table

*	1	*i*	*j*	*k*
1	1	*i*	*j*	*k*
i	*i*	-1	*k*	-*j*
j	*j*	-*k*	-1	*i*
k	*k*	*j*	-*i*	-1

4 Geometric Interpretation

A geometric interpretation was finally provided in 1799 by Caspar Wessel.

4.1 The Idea

Think of the standard number line, positively increasing to the right and negatively increasing toward the left, with zero in the middle. The vertical axis of the complex number plane, or imaginary axis, is orthogonal to the real axis at zero. So essentially, there exists a line perpendicular to the real number line at zero, which increases positively in the upward direction and negatively in the downward direction (see Figure 3).

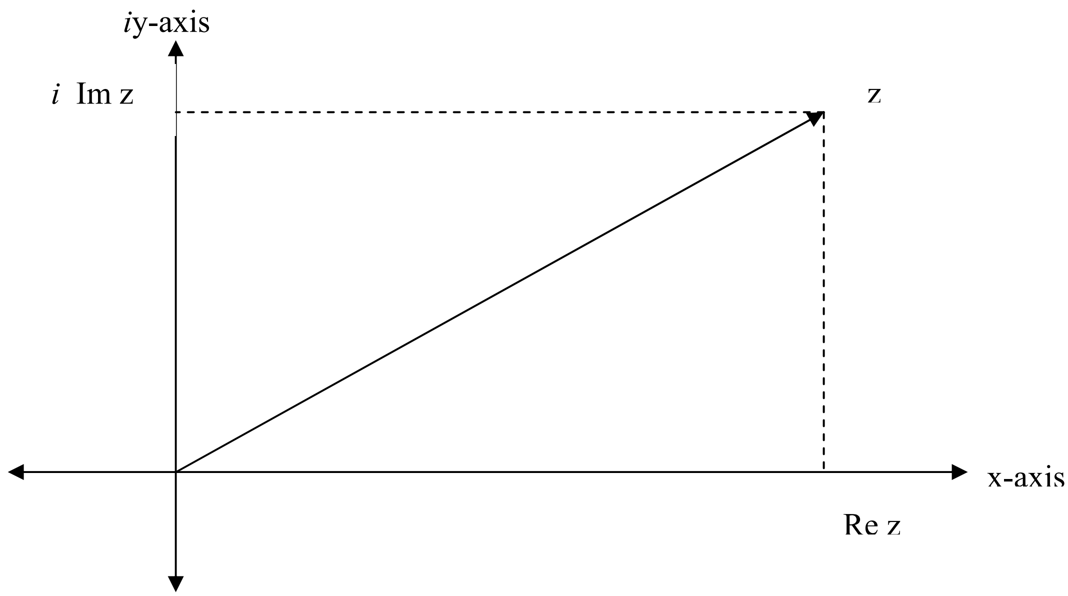
4.2 The Creators

Interestingly, the problem of the geometric interpretation of imaginary numbers was not solved by a professional mathematician, but by the surveyor Caspar Wessel from Norway in 1799. In fact it was possibly Wessel's occupation that aided in his breakthrough of the problem. With the assistance of the president of the science department of the Royal Danish Academy of Sciences, Wessel published a paper on his discovery. Sadly, the remarkable paper went unnoticed until 1895, almost 100 years after its publication.

Wessel's idea was subsequently rediscovered twice in the year 1806. The first of these rediscoveries was by Swiss-born mathematician Jean-Robert Argand, of whose background very little is known. 1806 found Argand in Paris working as a bookkeeper. Despite his non-mathematical origins, Argand published a small amount of pamphlets of his own work, upon which he failed to even print his name, on the geometry of imaginary numbers on a private press, a work that seemed fated to obsurity from the very beginning, even faster than Wessel's. This was surprisingly not the case.

Somehow, a copy of Argand's work found its way to prominent mathematicians Adrian-Marie Legendre, Francois Francais and his brother Jacques Francais. When Jacques Francais found out who the annonymous mathematician who wrote the work was, he declared Argand to be the first to have developed the geometric interpretation of imaginary numbers (neither of them had ever heard of Wessel)[1]. Though Argand's work was more noticed then Wessel's, their ideas continued to be obscure. The second rediscovery of Wessel's work was by French mathematicain, Abbé Adrien-Quentin Buée, and had a sort of mystical element to it. Like those that had tried before him, Buée's work went almost completely unnoticed. It wasn't until 1831 that the geometry of imaginary numbers was finally popularized 32 years after Wessel's initial discovery by German Carl Friedrich Gauss, a man thought to have been one of the three greatest mathematicians of all time. It was the

Figure 3: Imaginary Axis



work of Gauss that first began to expel the mysticism from the idea of imaginary numbers, mysticism Buée attempted to reinforce. Gauss gave imaginary numbers the "same full citizenship rights in mathematics as those enjoyed by real numbers." [3]

5 Formulas by Euler

Two formulas attributed to Euler are

$$e^{ix} = cosx + isinx$$

and

$$i^i = e^{-\frac{\pi}{2}}.$$

Interestingly enough, this last equation, Euler apparently didn't feel like proving. He basically just wrote it down and left it at that.

6 $e^{ix} = cosx + isinx$

6.1 Proof

Define a function f by $f(x) = \left(\frac{cosx+sinx}{e^{ix}}\right)$ The derivative of that function, by the quotient rule is

$$\begin{aligned} f'(x) &= \left(\frac{(-sinx + icosx)e^{ix} - (cosx + isinx)ie^{ix}}{(e^{ix})^2}\right) \\ &= \left(\frac{-sinxe^{ix} + icosxe^{ix} - icosxe^{ix} - i^2sinxe^{ix}}{e^{2ix}}\right) \\ &= \left(\frac{-sinx - i^2sinx}{e^{ix}}\right) \\ &= \left(\frac{-sinx + sinx}{e^{ix}}\right) \\ &= 0. \end{aligned}$$

Thus, $f(x)$ is a constant function, and we are able to set $f(x) = f(a)$ for any a in the real numbers. Therefore,

$$f(x) = f(0) = \left(\frac{cos(0) + isin(0)}{e^0}\right) = 1.$$

It follows that

$$\left(\frac{cosx + isinx}{e^{ix}}\right) = 1,$$

and therefore

$$cosx + isinx = e^{ix}.$$

6.2 Implications

The equation $e^{ix} = cosx + isinx$ provides alternative definitions to both the sine and cosine functions,

$$cosx = \left(\frac{e^{ix} + e^{-ix}}{2}\right)$$

and

$$sinx = \left(\frac{e^{ix} - e^{-ix}}{2i}\right).$$

There is a very special case of this equation, where $x = \pi$. The equation gives the famous Euler identity, an identity which contains what are thought by many in the mathematics world to be the five most important numbers in mathematics.

$$e^{i\pi} + 1 = 0$$

7 $i^i = e^{-\frac{\pi}{2}}$

7.1 Proof

Consider a circle of radius 1 centered at the origin,

$$x^2 + y^2 = 1.$$

To calculate the area of the first quadrant, which we already know to be $\frac{\pi}{4}$ using the area formula for a circle, we can take

$$\int_0^1 \sqrt{1 - x^2}dx.$$

By substituting $x = -iu$ and $dx = -idu$, we arrive at a new integral, keeping in mind the new limits of integration,

$$Area = \int_0^i \sqrt{1 - (-iu)^2}(-idu)$$

$$= -i\int_0^i \sqrt{1 + u^2}du$$

Using integral tables, we know:

$$Area = \int \sqrt{a^2 + x^2}dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\ln(x + \sqrt{a^2 + x^2}) + C.$$

Thus, assuming $a = 1$ and $x = u$ in the integral table formula,

$$Area = \frac{\pi}{4} = -i(\frac{1}{2}\,u\sqrt{u^2+1} + \frac{1}{2}\,\ln(u+\sqrt{u^2+1}))\Big|_0^i$$
$$= \frac{-1}{2}\,i\ln(i).$$

Therefore,

$$\frac{\pi}{4} = \frac{-1}{2}\,i\ln(i).$$

Performing a few arithmetic operations, we can conclude

$$\frac{-\pi}{2} = i\ln(i)$$
$$e^{\frac{-\pi}{2}} = i^i.$$

8 $i$nteresting Facts

There are some little known facts about imaginary numbers that some may find intriguing. One of these facts is that zero is the only number that is both real and imaginary. Also, strangely enough, the letter j is used as the imaginary unit in electrical engineering rather than i, due to the fact that i also denotes changing current.

8.1 Euler's Blunder

When imaginary numbers were first introduced, there was confusion about the fact that the equation

$$\sqrt{-1}^2 = \sqrt{-1}\sqrt{-1} = -1$$

was inconsistent with the algebraic identity

$$\sqrt{a}\sqrt{b} = \sqrt{ab}.$$

Euler actually used this identity incorrectly and argued

$$\sqrt{-1}\sqrt{-4} = \sqrt{4} = 2,$$

when in fact,

$$\sqrt{-1}\sqrt{-4} = i\sqrt{1}i\sqrt{4} = 2i^2 = -2.$$

This mistake prompted Euler to use the letter i in place of $\sqrt{-1}$.

9 Conclusion

Imaginary and complex numbers have effected both the fields of math and science, such as Calculus, Physics, Quantum Mechanics, and Control Theory. One of the first computers was even designed specifically to multiply and divide complex numbers. The initial discovery of imaginary numbers by Cardan and Tartaglia, though a mathematical breakthrough, did not prompt a wide acceptance of the idea. It was not until Gauss's geometric interpretation and Euler's extensive mathematics involving imaginary numbers that the idea of the numbers was widely accepted within the mathematic community. Imaginary numbers had a difficult and interesting beginning, but they have permanently made their place in the academic world.

References

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[3] Nahin, Paul J. *An Imaginary Tale: The Story of* $\sqrt{-1}$. Princeton: Princeton University Press, 1998.