Cuneiform and Our Understanding of Mathematics

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Abstract

This paper explains how our understanding of ancient mathematics was greatly augmented by the decipherment of the cuneiform script and the dead languages of Babylon.. . .

1 Introduction

Some of the earliest known languages were written in a form called cuneiform, and so the earliest known mathematical texts are written in cuneiform. These texts were written as long as 4000 years ago, and they were discovered after 1600. Because they are so ancient, the first linguists and historians to examine them were clueless as to what the writing meant. Once these ancient languages were understood better, ancient Mesopotamian civilizations could be examined much more closely. Now we have gained valuable information about early mathematics by translating these texts.

1.1 The Oldest Writings

Cuneiform is not a language but a style of writing, comparable to our alphabet which is also used (with minor alterations) in Dutch, German, Spanish and Latin. Cuneiform was written primarily in Mesopotamia from 3300 BCE to 75 CE; about a dozen languages were written using cuneiform, including the earliest known language–Sumerian, which has been dated back to 2800 BCE. Other examples of languages in which cuneiform writing was used are

Akkadian (also called Babylonian), Old Persian, Hittite, and Elamite.

In the 6th century BCE the Persian king Darius the Great commissioned a monument to be made on a mountain called Bisitun (or Behistun). The monument includes a relief sculpture of him stepping on the back of a defeated enemy, and cuneiform inscriptions in three languages: Akkadian, Elamite, and Old Persian. The third language, Old Persian, had never before been written with cuneiform script at that time.[1]

There are about 500,000 cunieform tablets in existence, and about 400 of them are strictly mathematical.[2] Many of these were only discovered in the 20th century. A few were found as early as 1602. With such an abundance of texts and the difficulties of deciphering the cuneiform script, our understanding of the cuneiform texts, especially the mathematics, has been a slow unfolding for the past four centuries. Some of these tablets were known to scholars for over two centuries before they could even translate a sentence of the inscriptions. There were several contributors that have enabled us to learn from these Babylonian texts.

1.2 Georg Grotefend (1775-1853)

Georg Grotefend was a high school teacher who had studied at the University of Gottingen in Germany. He had some knowledge of ancient inscriptions, but he was not an orientalist. He also had an interest in deciphering cryptograms.[3] He was described by his colleagues as an insatiable learner and researcher, sometimes studying from morning until night.

When Grotefend was twenty-seven, a friend asked him if it was possible to understand any cuneiform language, considering that the alphabet, vocabulary, and grammar were totally unknown.[1] Through his tenacity and analytical skills, Grotefend demonstrated just a few months later, in 1802, that it was indeed possible. He thought that certain Old Persian cuneiform inscriptions would include the names of kings Xerxes and Darius; Silvestre de Sacy had found this formula in several Middle Persian inscriptions.[3] Grotefend examined the names of the kings in other ancient languages, and correctly identified the Old Persian phrase, “Xerxes, great king, king of kings, son of Darius, great king, king of kings, son of Hystaspes.” From 1802 to 1805 he published several articles which were the first successful decipherment of any cuneiform writing.[1]

1.3 Henry Rawlinson (1810-1895)

Rawlinson was a British man whose life was marked by military pursuit and academic pursuit. From his youth he had found great motivation in both study and athletics. As a teenager, he travelled to India with the military. He equipped, trained, and led troops in India and Persia during the expansion of the British Empire. As a result of his involvement in the military in Asia, he gained great mountaineering skills, proficiency in the Persian language, and an enduring fascination with antiquities and Asian languages. This fascination eventually began to overshadow his pursuits of hunting and military action.[1]

1.3.1 Copying the Inscriptions at Bisitun

In the summer of 1836, he began studying the Bisitun inscription, which no one had been able to copy before because of its inaccessibility. His set of skills was perfect for the task. The inscription was 200 feet above the ground, and Rawlinson climbed the rock repeatedly without any scaffolding or a harness. (A French artist who made the ascent once in 1840 returned to the bottom "wounded, cut by the sharp angles of stones, completely torn and bloody."[1]) Once at the top, Rawlinson had to walk on a narrow cliff's edge, sometimes stacking one wooden ladder on another to reach certain parts of the inscription. It took many climbs, but he copied enough in one summer to be able to study the writing. He began to compare it to the Elwend inscriptions, which he had copied himself in 1835.[1]

1.3.2 Interpreting the Writing

In late 1836, Henry Rawlinson concluded, like Grotefend earlier, that the names of the kings must be present in the writing. After returning to Bisitun in 1837 to copy more of the Old Persian writing, he was able to make substantial progress. He methodically transliterated the signs into Roman characters, and then translated these into English sentences. By 1840 he had translated several paragraphs of the inscription at Bisitun, but during that time he was working in isolation at an arsenal in Tehran. He did not have access to the current European scholarly work, but he had better resources for cuneiform study than the other European researchers.

When Henry Rawlinson was twenty-seven years old, on New Year's Day 1838, he sent a translation of two paragraphs of the Bisitun inscription to the Royal Asiatic Society in London. Much of this translation work was accomplished through comparison of related words in the ancient languages Avestan and Sanskrit. Throughout that year, Rawlinson was in correspondence with the Society in London, and the Societe Asiatique in Paris. One message could take two months to be received, but the researchers in Europe were astonished and excited about Rawlinson's work. They sent him the most current research on cuneiform, including cuneiform alphabets which were suggested by Saint-Martin and Burnouf. Their work did not completely coincide with Rawlinson's, and there was a dispute between Rawlinson and Burnouf; ironically, the correspondence he received in this dispute greatly increased his confidence in his way of interpreting Old Persian cuneiform. After he studied Burnouf's commentary on the Avestan language, Rawlinson became the first person to confront Old Persian "as a language, and not just a cuneiform script."[1]

2 Numbers and Notation of Akkadian Mathematics

Before exploring any of the mathematics in Akkadian tablets, it is necessary to have some understanding of their numeral system. There are also unique features to the notation used in cuneiform mathematics.

2.1 The Sexagesimal Place Value System

Babylonian scribes used sexagesimal numbers instead of decimal numbers (base 60 rather than base 10). For the decimal place value system, any integer a may be written as

$$a = a_{n-1} \times 10^{n-1} + ... + a_2 \times 10^2 + a_1 \times 10^1 + a_0$$

and we shorten the form of the number a to $a_{n-1}...a_2a_1a_0$. Similarly, for the sexagesimal system any integer a may be written as

$$a = a_{n-1} \times 60^{n-1} + ... + a_2 \times 60^2 + a_1 \times 60^1 + a_0$$

and we shorten the form of the number a to $a_{n-1}...a_2a_1a_0$. That is the basis of the sexagesimal place value system. One sexagesimal place has a value between 00 and 59 instead of between 0 and 9. As a result, one sexagesimal place has two digits in Hindu-Arabic numberals. 1000 in decimal notation would be written as 16 40 in sexagesimal notation because

$$16(60) + 40 = 960 + 40 = 1000.$$

Also note that fractions were in use. Briefly, here are some examples of decimal fractions converted to sexagesimal:

$$\begin{array}{l} \frac{1}{6} = \frac{10}{60} =; 10 \\ \frac{1}{5} = \frac{12}{60} =; 12 \\ \frac{1}{4} = \frac{15}{60} =; 15 \\ \frac{1}{3} = \frac{20}{60} =; 20 \\ \frac{1}{2} = \frac{30}{60} =; 30 \end{array}$$

Modern writers use the sexagesimal semi-colon to mark the separation between the 1s place and the $\frac{1}{60}$ place; it is analogous to a decimal point.

The Babylonians also used numbers with both an integer part and a fraction part[4], such as:

$$\frac{3}{2} = 1 + \frac{30}{60} = 1; 30$$

2.2 Conversions

With a basic understanding of the sexagesimal place value system, we can convert from sexagesimal to decimal or from decimal to sexagesimal with little difficulty. Interestingly, in ancient Mesopotamian civilizations, these conversions were a common practice. The educated scribes used the Sumerian-Akkadian cuneiform numbers, which were sexagesimal, while most Semitic people used their own words for decimal numbers. One clay tablet, MS 3970, contains an unfinished conversion table in which multiples of 100 are written in sexagesimal forms: $11\ 40 = 700, 13\ 20 = 800$, etc.[4]

To convert 10000 from decimal to sexagesimal, the following method is used:

$$10000/60 = 166.66.../60 = 2.77...$$

So the first hexagesimal place is 2. (Representing 2×60^2.) Subtract 2 and multiply by 60 to find the value in the 60s place.

$$2.77... - 2 = .77... \times 60 = 46.66...$$

The second hexagesimal place is 46. Subtract 46 and multiply again.

$$46.66... - 46 = .66... \times 60 = 40$$

The 1s place contains 40. So 10000 in decimal is 2 46 40 in hexagesimal. To check our results, we convert back to decimal:

$$\begin{gathered} 2\ 46\ 40 = 2 \times 60^2 + 46 \times 60^1 + 40 \times 60^0 \\ 2 \times 60^2 = 7200 \\ 46 \times 60 = 2760 \\ 40 \times 1 = 40 \\ 7200 + 2760 + 40 = 10000 \end{gathered}$$

2.3 Cuneiform Mathematical Notation

In cuneiform mathematics there was no marker to separate whole numbers from the fractions. Modern historians use the “sexagesimal semi-colon” for this purpose but there is no such separator on the tablets. So the reader has to be able to distinguish between 35 13 and $35; 13$ by context.

Another problem encountered by scholars today is that the Old Babylonians lacked final zeros and initial zeros in their notation. So a number like 42 00 00 could not be distinguished from 42 00, 42, or even ;42.

There were no internal zeros either, but scribes could express the internal zero with a gap. In MS 2049, the number 1 19 00 44 26 40 is written as

$$1\ \ 19\qquad 44\ \ 26\ \ 40.$$

(In the blank space the scribe had written 44, and he or she erased this and left the space to indicate a value of zero.)

For these reasons, a sexagesimal number in a Babylonian cuneiform text is said to have a “relative” value. The absolute or intended value is only determined in relation to other numbers on the same tablet. To the modern student, this notation (or lack thereof) seems impractical, but the Old Babylonians were able to perform some calculations that have surprised the researchers of the 20th century. Joran Friberg[4] points out also that “it can be very convenient to be able to count with sexagesimal numbers without having to bother about the absolute value of the numbers . . . ”

3 Operations

We only have examples of a few mathematical operations being worked out in cuneiform mathematics. Some operations may have been performed mentally or possibly using tools.

3.1 Addition

There is no example in all of the known cuneiform tablets of complicated addition performed step by step. The answer to any addition problem is always given directly. There are a few reasons why this would be, one being that addition is the easiest operation to perform mentally. For larger numbers, they may have used a counting board which could produce the answers for them, or a separate tablet which could be erased and used again. The clay tablets used for writing were difficult to produce and could not be used frivolously.[4]

Adding in another base in a place value system is only different during carrying over; in base 10, the digits are carried over when they add up to 10, and in base 60 they are carried over when they add up to 60. Here is a short example:

$$\begin{array}{r} 55\ 40 \\ \underline{+13\ 40} \\ 1\ 09\ 20 \end{array}$$

Here is how the answer was obtained in this example:

$$\begin{gathered} 40 + 40 = 80 \\ 80 - 60 = 20 \end{gathered}$$

So 20 is in the 1s place and 1 is carried over, because $60 = 1\ 00$.

$$\begin{gathered} 55 + 13 + 1 = 69 \\ 69 - 60 = 9 \end{gathered}$$

09 is the 60s place and 1 is carried over. The result is 1 09 20.

3.2 Subtraction

Again, because a place value system is in use, subtraction in sexagesimal is not totally different from subtraction in decimal numbers. The differences are the use of double-digits, and borrowing sixties instead of tens. This is illustrated in the following example:

$$\begin{array}{r} \underline{60} \\ 35\ 40 \\ \underline{-23\ 50} \\ 11\ 50 \end{array}$$

Because 50 cannot be subtracted from 40, 1 is pulled out of the sixties place and used as if it were 60 in the ones place. So 40 is added to 60, and 50 is subtracted, thus the result 50. Then, because one was borrowed from the sixties place, one is subtracted, which is why the result is 11 50 and not 12 50. For long computations the process would be the same.

3.3 Multiplication

There are three Old Babylonian tablets with elementary multiplication: MS 2728, MS 2729, and MS 3944.[4] As it is with addition, there are no computations written that show what kind of algorithm the writer used to find the product of two numbers. Ideas about the multiplication methods of the Babylonians may be considered speculation, but it is interesting to see the different paths to a solution which are allowed because of their numeral system.

This is where one of the advantages to a relative place value notation may be seen. 30 may be thought of as ;30 (equal to $\frac{1}{2}$), 20 may be thought of as ;20 (equal to $\frac{1}{3}$), etc. Here are a few multiplications which are simplified by this method of thought:

$$30 \times 2\ 32 = \tfrac{2\ 32}{2} = 1\ 16\ (00)$$
$$20 \times 57 = \tfrac{57}{3} = 19\ (00)$$
$$40 \times 27 = \tfrac{2\times 27}{3} = \tfrac{54}{3} = 18\ (00)$$

(Recall that an answer such as 18 for the third problem is not correct in terms of modern sexagesimal mathematics, but the equivalent answers in cuneiform would have been considered correct because of the absence of final zeros in their system.)

There are other methods of finding products that differ little from modern methods in decimal notation. For example, here is a problem solved through the addition of "partial products":
$25 \times 48 = (20 + 5) \times 48 = 20 \times 48 + 5 \times 48 = 16\ (00) + 4\ (00) = 20\ (00)$

4 The Language of Cuneiform Mathematics

When examining cuneiform mathematics, we have to adjust our mindset to the language as well as the notation. The vocabulary used in Akkadian geometry problems is unique to the Akkadian language, so terms are not used in the same manner that they are used in English.

4.1 Terminology

The terminology that we use today for geometry has been determined largely by the cultures that came before us. In Mesopotamia thousands of years ago, the Babylonians were influenced by the Sumerians, but as far as we know the Sumerians may have generated much of their own terminology. Generally, the primary difference to be noted between today's mathematics and the mathematics of the Old Babylonian period is the modern drift towards abstraction.

Today the word "angle" does not sound abstract–yet an "angle" is not a tangible object, but a measure of space. There is no Akkadian word, then, "right angle" or for "angle." This is problematic when trying to translate the geometric terms from cuneiform to modern languages. Our word triangle means "three-angled", but the corresponding word in Akkadian cuneiform has no reference to number of angles or corners. The word used for triangle is *santakkum* and means "wedge".[4]

Other examples of words that are problematic to use in translations of Akkadian are diagonal, pentagon, hexagon, and heptagon. The suffix *-gon* means corner or angle, something which the Babylonians had no reference to. It is a daunting task, then, to accurately translate cuneiform mathematics problems without using the more familiar terms which were derived primarily from Latin and Greek.

Some linguists believe that the drift over time from concrete to abstract words is one of the largest driving forces behind language change.[5] It is no surprise, then, what a literal translation of cuneiform shows: rather than a specific mathematical word for "add" which has become somewhat abstract to us, the Akkadian word is literally "heaped." Square in cuneiform would be literally translated as "sameside" (i.e. a shape with all sides of the same length).

4.2 Translations and Transliterations

Various methods have been used to downplay the difficulty of expressing cuneiform mathematics to the reader who has minimal familiarity with cuneiform. Any translation will have its disadvantages to either the experienced scholar or the less experienced student. A compromise must be made between the two extremes of attempting to express exactly what was meant in the original language and creating a translation that is accessible to a reader with no previous knowledge of the original language. The method used by Otto Neugebauer, Francois Thureau-Dangin, and Abraham Sachs in the first half of the twentieth century was essentially the same method used by Henry Rawlinson in the 1830s and 1840s. The first step was to transliterate the cuneiform into a modern alphabet. The second step jumped from this transliteration to a translation into standard English.

Hoyrup was the first to see the advantages of using a one-to-one translation, using different words in the translation to correspond to different Babylonian words, even where the modern language does not presuppose the use of different terms. Since 1982 he has been publishing his discoveries regarding geometric models in Old Babylonian mathematics.

Recently Joran Friberg has developed what he calls "conform" transliterations and translations. The objective in a conform transliteration is to mimic the original pronunciation as closely as possible; the objective in a conform *translation* is to represent as closely as possible the non-mathematical meanings of the words in the original texts.

An example of a conform transliteration and translation from Friberg's work is given below for the sake of example:

> a.sà 2 tés.a.sì gar.gar-*ma* 21 40
> tés.a.sì gar.gar-*ma* 50 /
> tés.a.sì.mes en.nam
> The fields of 2 samesides (I) heaped, then 21 40.
> The samesides (I) are heaped, then 50.
> The samesides (are) what?[4]

This is closer to the original language, of course, but it requires some familiarity with the terms used by the Babylonians. The word for "sameside" can mean either a square or a side of a square. A standard English translation would be:

> The areas of 2 squares added together equals 21 40. The sides of the squares added together equals 50. What are the sides of the squares?[4]

5 Plimpton 322

Plimpton 322 is a cuneiform tablet that is kept in Columbia University's Plimpton Collection of rare antiquities. The tablet was probably created between 1900 and 1600 BCE.[2] When this tablet was first translated by researchers, no one discovered the interesting relationships between the numbers. It could have appeared to be only a catalogue created for a merchant, but it is actually a mathematical table involving right triangles with sides of integer values. In 1945, Neugebauer and Sachs published their findings concerning this tablet. They were the first to explain how the numbers in the table are related to right triangles and the Pythagorean theorem. Since that publication, Plimpton 322 has been one of the most examined Babylonian mathematical tablets.[2]

5.1 The Table

The table in Plimpton 322 contains three columns of numbers, shown in Table 1. The last column only numbers the lines of the tablet, but the first two numbers in any row are two of the three numbers in a Pythagorean triple. A Pythagorean triple is a set of three positive integers which can be the sides of a right triangle, like (3,4,5).

In Table 1, the first nine rows of the numbers in Plimpton 322 are shown, converted to decimal numbers.

Table 1: Plimpton 322

119	169	1
3367	4825	2
4601	6649	3
12709	18541	4
65	97	5
319	481	6
2291	3541	7
799	1249	8
481	769	9

5.2 Pythagorean Triples Given Parametrically

A Pythagorean triple is called a primitive Pythagorean triple if the three numbers have no integral factor in common (other than one). For example, (6, 8, 10) is a Pythagorean triple, but it is not primitive because all three numbers are divisible by two. (3, 4, 5) is an example of a primitive Pythagorean triple.

Primitive Pythagorean triples may be found using the parametric representations:

$$a = 2uv$$
$$b = u^2 - v^2$$
$$c = u^2 + v^2$$

If u and v are relatively prime, of different parity, and u is greater than v, then a primitive Pythagorean triple may be obtained by plugging u and v into the equations above.

All of the Pythagorean triples in Plimpton 322 are primitive except for two of them. This means that the Babylonians may have known some form of the parametric representations above. It is possible that the scribe who wrote Plimpton 322 knew this method of calculating primitive triples.

6 Conclusions

In 1602, the Europeans discovered cuneiform inscriptions. Two hundred years later, Grotefend deciphered several words of Old Persian; in 1838, Rawlinson published paragraphs of translation. About a hundred years after that, in 1945, Neugebauer and Sachs published their unexpected discoveries about Pythagorean triads in a Babylonian tablet; and in 2007 Friberg has published his work on dozens of tablets (some previously unpublished) showing many of the complexities of the cuneiform tablets which have only been revealed through careful study.

As research has progressed, we have only gained respect for the intellectual acheivements of ancient civilizations. It is ironic how difficult it has been to understand civilizations that were once conceived to be simple-minded or even savage. The findings of the twentieth century have shown us that the Babylonians were more sophisticated than previously thought, at least in their mathematics. Their prose algebra was well-developed by 2000 BCE, and they were able to obtain many practical geometric solutions.[2] The complexity of their computations is continuing to surprise scholars today.

7 References

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