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Abstract: This paper will discuss the works and influences of Diophantus of Alexandria. Through his puzzling details of life, his surviving works, his advancement in algebraic notation, and his influence on mathematicians, such as Fermat, Diophantus made a great impression upon mathematics.

The Life and Works of Diophantus of Alexandria

Diophantus of Alexandria is known as the "father of algebra". There is little detail known about his life with exception of his works. Only two of his works have survived and only pieces of them remain today. His work with algebra includes number theory and mathematical notation. His work influenced many others and was crucial in the development of algebra in Europe in the late sixteenth century through the eighteenth century. He was famous with Arab mathematicians and influenced their work greatly. Diophantus' work helped to create a foundation for work on algebra. Diophantus is known as the last, and one of the most successful, Greek mathematicians. The debate on the dates of his life, his work *Arithmetica*, his advancements in algebraic notation, and his work *Porisms*, make Diophantus an extremely interesting Greek mathematician.

DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.
AVCTORE CLAVDIO GASPARE BACHETO MEZIRIACO SEBVSIANO, V.C.
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

While there is much debate about his exact dates of life, it is generally assumed that Diophantus flourished around 250 A.D. According to Heath, one of the reasons why we have such a hard time placing Diophantus in a time period is because there was another Diophantus who lived in the time of the Emperor Julian in A.D. 361-363. There is, however, a way to establish limits on his possible dates of life. In

Diophantus' book *On Polygonal Numbers*, he quotes a definition from Hypsicles. This means that he wrote it later than 150 B.C. Theon of Alexandria, who worked from 365 A.D. to 372 A.D., quoted Diophantus in his works, and therefore, Diophantus must have written before 350 A.D. A letter of Psellus exists in which both Diophantus and Anatolius are mentioned as writers on the Egyptian method of reckoning. "Diophantus dealt with it more accurately, but the very learned Anatolius collected the most essential parts of the doctrine as stated by Diophantus in a different way and in the most succinct form, dedicating his work to Diophantus" (Heath, "Diophantus" 2). This implies that Diophantus and Anatolius were contemporaries and since Anatolius wrote around 278-279 A.D., the general agreement is that Diophantus flourished around 250 A.D. Diophantus was not mentioned by Nicomachus (c. 60 – c. 120), Theon of Smyrna (c. 70 – c. 135), or Iamblichus (c. 250 – c. 325) and therefore, 250 A.D. is a reasonable date for his works.

"An arithmetical epigram on his age is attributed to Metrodorus" (Gow 100). Mentioned in the Anthology, this mathematical problem describes Diophantus' life through major events.

> "Diophantus passed $1/6$ of his life in childhood, $1/12$ in youth, and $1/7$ more as a bachelor; five years after his marriage was born a son who died four years before his father, at half his father's age" (Cajori 60).

This information leads to the equation that $1/6\,x + 1/12\,x + 1/7\,x + 5 + 1/2\,x + 4 = x$. When solved, this equation gives $x = 84$, implying that Diophantus was 84 when he died. "It is clear that that the epigram was written, not long after his death, by an intimate personal friend with knowledge of and taste for the science which Diophantus made his life-work" (Heath, "Diophantus" 3).

Diophantus presented new ideas and approaches to mathematics, and due to this, his work resembles nothing of other Greek mathematics. "If his works were not written in Greek, no one would think for a moment that they were the product of a Greek mind" (Cajori 60). Algebra was much like an unknown branch of mathematics to the Greeks, except for Diophantus.

The Arithmetica, concerned with the theory of calculation with numbers, is one of the most influential works in algebra (Gittleman 82). It was written in 13 books. This work is characterized by its degree of mathematical skill and that it represents a new branch in mathematics unlike any traditional Greek work (Boyer 180). Diophantus' *Arithmetica* is the earliest piece on algebra in existence today. Only 6 of the 13 books have survived, although some divide them into 7 books, while others include the piece *On Polygonal Numbers* as book 8. The missing 7 books were apparently lost at an early date. One theory is that Hypatia's commentary on *The Arithmetica* only consisted of the first six books and therefore the others were first forgotten, and then lost.


Arithmeticorum Liber II. 61

QVÆSTIO VIII.

OBSERVATIO DOMINI PETRI DE FERMAT.

QVÆSTIO IX.

Diophantus only dealt with positive rational numbers, but was the first to accept fractions as solutions. "He is the first Greek to whom a fraction was a number and not a ratio" (Gow 112). When he came upon a solution that was a negative number or an irrational number, he thought it was absurd, because he did not see any practical use for a negative solution.

The Arithmetica consists of about 130 problems, giving numerical solutions to determinate and indeterminate equations, each solved by a special method. Only the first book is about the solutions of determinate equations, while the rest of the books deal with solutions to indeterminate quadratic equations and of simultaneous equations. He was more concerned with ways to divide numbers rather than the nature of numbers. "Diophantus was only concerned with integral or fractional solutions…which made his problems much more difficult than modern algebra problems" (Gittleman 82).

He begins his *Arithmetica* with a letter to Dionysius, and a list of eleven definitions, in which he states the definitions' symbols and rules. The letter begins like this:

> "Knowing that you are anxious to learn the solution of arithmetical problems, I have tried to systemize the method, beginning from the foundations of the matter. You will think it hard before you get thoroughly acquainted with it" (Gow 108).

After this letter, Diophantus gives definitions of what he called different "species" of numbers. These were the various powers of the unknown from the second to the sixth, the unknown quantity, and units. After listing these species, Diophantus listed the notation used for each. It is in this work that Diophantus is credited with the first algebraic notation.

> "It is from the addition, subtraction or multiplication of these numbers or from the ratios which they bear to one another or to their own sides respectively that most arithmetical problems are formed" (Heath, "Diophantus" 130).

He then states that the work will be written in thirteen books and begins his problems. Book one consists of 39 problems. After many of the problems, Diophantus lists necessary conditions.

> For example, problem 9 states:
>
> From two given numbers to subtract the same (required) number so as to make the remainders have to one another a given ratio.
>
> *Necessary condition*. The given ratio must be greater than the ratio which the greater of the given numbers has to the lesser.
>
> Given numbers 20, 100, given ratio $6 : 1$.
>
> Required number x.
>
> Therefore $120 - 6x = 100 - x$, and $x = 4$ (Heath, "Diophantus" 133).

Diophantus solved many determinate equations of the second degree but never went through the entire process of solving quadratic equations. He wrote the quadratic equation in a way so that all terms were positive. This makes logical sense because he only dealt with positive rational numbers. "There were three cases of equations with a positive root: $ax^2 + bx = c$, $ax^2 = bx + c$, $ax^2 + c = bx$, each case requiring a rule slightly different from the other two" (Cajori 61). Diophantus stops once he attains a single solution. It is likely that he was aware that it was possible to obtain two positive rational solutions. It is surprising then, that he stopped after reaching just one.

Among all the determinate problems in *Arithmetica*, book I, there is actually one indeterminate problem. This problem, number 14, is as follows:

> Find two numbers such that their product has a given ratio to their sum. One of the two numbers must be greater than the number representing the ratio.
>
> Let the product be 3 times the sum, and let one of the numbers be *s*. The other must be greater than 3, let it be 12.
>
> The product is $12s$, the sum $12 + s$.
>
> Therefore $12s$ equals $3s + 36$, and $s = 4$.
>
> The two numbers are 4 and 12.
>
> In the solution procedure, the indeterminate problem a · b = p · $(a + b)$ with $p = 3$ is made determinate by arbitrarily assuming that one of the numbers is 12 (greater than $p = 3$). The diorism is not explained, but since
>
> $$a \cdot b = p \cdot (a + b) \Rightarrow \left(a - {}^{p}/_{2}\right) \cdot \left(b - {}^{p}/_{2}\right) = sq. {}^{p}/_{2}$$
>
> It follows that one of $a - {}^{p}/_{2}$ and $b - {}^{p}/_{2}$ must be greater than ${}^{p}/_{2}$
>
> (Friberg 331).

Diophantus was the first to introduce a form of algebraic notation. There are three stages in the evolution of algebra. The first being rhetorical algebra, the second being syncopated algebra, and the third is symbolic algebra. Diophantus falls under the second category for his system of notation. He defines the unknown quantity as “containing an indeterminate or undefined multitude of units” and then stating that it is simply called number and is denoted by a particular sign (Heath, “History” 456). Diophantus had only a symbol for one unknown but did use his notation to express different powers of the unknown, up to the sixth power. He also denoted the reciprocals of the powers of the unknown.

He merely used juxtaposition for addition, and he used a symbol for subtraction. Diophantus had no sign for multiplication because it wasn’t necessary due to the fact that his coefficients were all numbers or fractions. Since Diophantus did not have a sign for plus, he had to write all of the positive terms together and then followed them by the negative terms. He had only a few symbols, and sometimes he even ignored these he had and just described the operation in words. In addition to the definition he listed for subtraction was the following statement: "a wanting multiplied by a wanting makes a forthcoming; and a wanting multiplied by a forthcoming makes a wanting" (Heath, "Diophantus" 460).

The form of Diophantus' notation limits him to only one unknown. He basically needed to perform all of his eliminations at the beginning of the problem to get all of the terms written in terms of the only one unknown. When attempting to solve an indeterminate problem that would lead to an indeterminate equation with two or three unknowns, Diophantus must assume particular numbers to make the problem determinate. "The particular devices by which he contrives to express all his unknowns in terms of one unknown are extraordinarily various and clever" (Heath, "Diophantus" 461).

Though it did not survive to today, Diophantus' work *Porisms* was referenced three times throughout the *Arithmetica*. It is believed that this work was entirely devoted to the discussion of general properties of numbers. The three propositions Diophantus cites are as follows:

1. If $x + a = m^2$, $y + a = n^2$, and $xy + a = p^2$, then $m = n + 1$.
2. If three numbers x^2, $(x+1)^2$, $4x^2 + 4x + 4$, be taken, the product if any two plus their sum, or plus the remaining number, is a square.
3. The difference between two cubes may be resolved into the sum of two cubes. (Gow 120-1)

Before all of these propositions listed in the *Arithmetica*, he states that he find them in the *Porisms*. He also states many other propositions without stating that they are contained within the *Porisms*. These other propositions fall into two categories. The first category contains identities. For example,

$(x^2 + y^2) \pm 2xy$ is always a square (Gow 121).

The second category contains general propositions regarding the breaking up of numbers into the sum of multiple parts. For example,

> Every square number may be resolved into the sum of two square numbers in an infinite number of ways (Gow 121).

All of these propositions are in general form. It is suggested that Diophantus should have taken this route when writing the *Arithmetica* but he didn't.

Diophantus was extremely clever in the way he was able to reduce every problem to an equation which he knew he could solve. "The most common and characteristic of Diophantus' methods is his use of tentative assumptions which is applied in nearly every problem of the later books" (Gow 116). This method is for assigning a value to the unknown in a way that satisfies only one or two of the necessary conditions, and in failing the remaining conditions, makes what is needed to be clearly seen. Also, Diophantus was clever in using his symbol for the unknown in different senses. For example,

> Find 3 numbers, such that their sum is a square and that the square of any one of them plus the following number is a square.
>
> The 3 numbers are first taken as $x - 1$, $4x$, and $8x + 1$,
>
> where $(x+1)^2 + 4x$ and $(4x)^2 + 8x + 1$ are both square numbers.
>
> Two conditions are thus satisfied. But the sum of all three numbers, $13x$, must be a square.
>
> Take $13x$ equal to x^2 with some square coefficient, $169x^2$.
>
> Then $x = 13x^2$. A new use of x is thus introduced and $13x^2$ is substituted for the original x, the numbers now being $13x^2 - 1$, $52x^2$and $104x^2 + 1$.
>
> A fourth condition remains, that $(104x^2 + 1)^2 + (13x^2 - 1)$ shall be a square number.
>
> Diophantus then takes this expression equal to $x^2(104\mathrm{x} + 1)^2$, finds $x = \frac{55}{52}$, and substitutes this value in the expression (Gow 117).

Diophantus' methods of solution were unique and clever, but since each problem was to be solved in a particular way, and there was no general rule to follow for any problem, it has been said that a person could solve 100 Diophantine equations and not know how to solve the 101st.

For example,

> Express a given number which is the sum of two squares as the sum of two other squares.
>
> Solution. Let the given number be 13, the sum of the squares of 2 and 3. Let the sides of the two other squares be $s + 2$ (the 2 is chosen to match the given number 2), and $2s - 3$ (the 3 is chosen to match the given 3, while the 2 is arbitrary). Thirteen is to be the sum of these two squares also, so
>
> 13 = $(s+2)^2 + (2s-3)^2$
>
> $13 = 5s^2 - 8s + 13$
>
> $5s^2 = 8s$
>
> S = $\frac{8}{5}$
>
> Notice how Diophantus cleverly chose the unknowns to obtain an easy equation to solve and to insure a rational solution. Substituting for s, the squares are
>
> $$(s+2)^2 = \frac{324}{25}$$
>
> And
>
> $$(2s-3)^2 = \frac{1}{25}$$
>
> Their sum is 13. (Gittleman 82).

While Diophantus is known as the father of algebra, Fermat is known as the father of the modern theory of numbers. Fermat was so secretive in his work that he usually only made known his results, and kept his methods to himself. Fermat owned a copy of Bachet's *Diophantus*, in which he wrote many notes in the margins. Fermat's son published a new version of this book in 1670, which included Fermat's notes. The following theorem is found in Fermat's marginal notes:

1. $x^n + y^n = z^n$ is impossible for integral values of $x, y,$ and z, when $n > 2$.

   This famous theorem was appended by Fermat to the problem of Diophantus II, 8: “To divide a given square number into two squares.” Fermat’s marginal note is as follows: “On the other hand it is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or generally any power except a square into two powers with the same exponent. I have discovered a truly marvelous proof of this, however the margin is not large enough to contain” (Cajori 168).

This apparent proof was never found and it is believed that Fermat never actually had a proof. This is known as Fermat’s Last Theorem. This led to tremendous advances in number theory and the study of Diophantine equations. The theorem went unproven for many centuries. Andrew Wiles is given credit for the proof of this theorem, which he published in 1994, after working on it for 7 years. Without Diophantus’ work, Fermat would never have studied his theorems. Much is owed to Diophantus for work in algebra, which spurred many advances in number theory.

Although most of his work does not survive today, Diophantus was an extremely influential mathematician. His work was original, clever, and helped lay a basis for algebra as we know it today. “With Diophantus the history of Greek arithmetic comes to an end” (Gow 121). Through his notation, unique methods, and recognition of fractions as rational numbers, Diophantus can truly be called the “father of algebra”.

Works Cited

Boyer, Carl B. A History of Mathematics. New York: John Wiley & Sons, Inc., 1991.

Cajori, Florian. A History of Mathematics. New York: Chelsea Publishing Company, 1980.

Friberg, Joran. Amazing races of a Babylonian Origin in Greek Mathematics. Singapore: World Scientific Publishing Co. Pte. Ltd., 2007.

Gittleman, Arthur. History of Mathematics. Columbus: Charles E. Merrill Publishing Co., 1975.

Gow, James. A Short History of Greek Mathematics. New York: G. E. Stechert & Co., 1923.

Heath, Thomas L. Diophantus of Alexandria: A Study in the History of Greek Algebra. New York: Dover Publications Ins., 1964.

---. A History of Greek Mathematics: Volume 2.Oxford: University Press, 1965.