

# **A History of the Famous Three Problems and Zeno's Paradoxes**

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May 30, 2009

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Evolution of Mathematics

T-Th 8-9:20

## Abstract:

There were three famous Greek Problems proposed in the Hellenic period, along with paradoxes proposed by Zeno. This paper discusses these problems and provides a brief history on the era.

## Introduction of the Three Famous Problems

The Hellenic period is characterized by the Hellenistic civilization of the ancient Greek world, and the classical Greek era following. The term ‘Hellenistic’ is derived from “\_\_\_\_\_” (Héllén), which is the Greek’s traditional name for themselves. It was coined by historian Johann Gustav Droysen to refer to the spreading of Greek culture and colonization over the non-Greek lands that were conquered by Alexander the Great in the 4<sup>th</sup> Century BC.(1) During the first 300 years of Greek mathematics, there were 3 important and distinct lines of development. First, there was the material that was eventually organized into Euclid’s Elements. Next, were developments connected with infinitesimals and with limit summation processes. Third was the development of higher geometry, which is the geometry of curves other than the circle and straight lines on surfaces that do not include the sphere or plane. This last development on higher geometry originated in continued attempts to solve the well known “Famous Three Problems.”(2)

“There are, we say, three types of problems in geometry, the so called “plane”, “solid”, and “linear” problems. Those that can be solved with straight line and circle are properly called “plane” problems, for the lines by which such problems are solved have their origin in a plane. Those problems that are solved by the use of one or more sections of the cone are called “solid” problems. For it is necessary in the construction to use surfaces of solid figures, that is to say, cones. There remain the third type, the so-called “linear” problem. For the construction in these cases curves other than those already mentioned are required, curves having a more varied and forced origin and arising from more irregular surfaces and from complex motions.”-Pappus(3)

These Three Famous Problems proposed by the Greeks were:

1. The duplication of the cube, or the problem of constructing the edge of a cube having twice the volume of a given cube.
2. The trisection of an angle, or the problem of dividing a given arbitrary angle into three equal parts.
3. The quadrature of the circle, or the problem of constructing a square having an area equal to that of a given circle.

These three problems also came with a set of rules that were actually self-imposed more than 2000 years after the problems were created. It wasn't until about the nineteenth century when mathematicians stated that the problems needed to be solved with only an unmarked straightedge and a compass(2).

#### Problem #1, Squaring the Circle

The most famous of the three problems is the problem of squaring the circle, also known as the quadrature of the circle. This problem originated because it was easy to cut an angle into even halves, but it was hard to cut it into thirds. Of course angles of 45, 72, 90, and 180 were possible, but any number that could not be divided equally into thirds was an issue.

One of the oldest surviving mathematical writings is the Rhind papyrus, named after the Scottish Egyptologist Henry Rhind who purchased it in Luxor in 1858. This scroll was written around 1650 BC by the scribe Ahmes. However, Ahmes had copied his from the original. It is estimated that the original papyrus was developed around 1850 BC, but some experts believe that it can actually be based on a work dating back to 3400 BC.(3)

In his Rhind papyrus, Ahmes gave a rule to construct a square of area nearly equal to that of a circle. Ahmes cut  $\frac{1}{9}$  off the circle's diameter and constructed a square on the remainder. (He

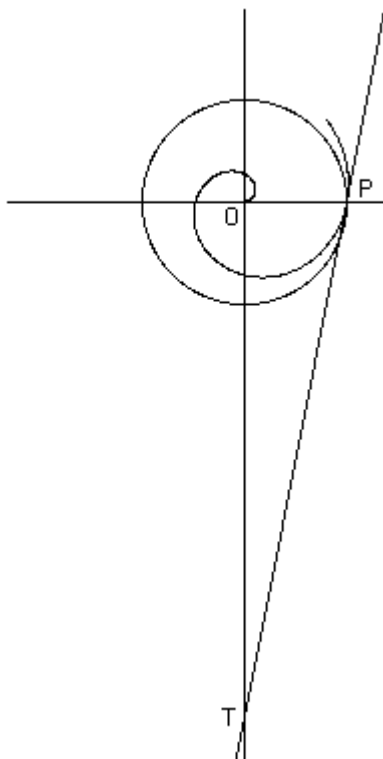
used what was left to form the square). Although this is not really a geometric construction, it does show that the problem of constructing a square of area corresponding to a value of 3.1605, rather than 3.14159, for  $\pi$ .(3)

Though many tried to solve this problem, Hippocrates was the first to use a plane construction to find a square with area equal to a figure with circular sides. He squared lunes, (which are parts of a plane surface bounded by two intersecting arcs or of a spherical surface bounded by two great circles )(10), however he did not show that every lune he used could be squared. In his work, there was no possible way for him to show that the lunes he used could be squared by plane methods, otherwise he would have solved the squaring of the circle problem.

The next great contributor to this problem was Archimedes, with his use of the spiral.

“If a straight line drawn in a plane revolves uniformly any number of times about a fixed extremity until it returns to its original position, and if, at the same time as the line revolves, a point moves uniformly along the straight line beginning at the fixed extremity, the point will describe a spiral in the plane.”-Archimedes, *On Spirals*.

To square the circle Archimedes gives the following



construction. Let  $P$  be the point on the spiral when it has completed one turn. Let the tangent at  $P$  cut the line perpendicular to  $OP$  at  $T$ . Then Archimedes proves in Proposition 19 of ‘On spirals’ that  $OT$  is the length of the circumference of the circle with radius  $OP$ . Now it may not be clear that this is solved the problem of squaring the circle but Archimedes had already proved as the first proposition of ‘Measurement of the circle’ that the area of a circle is equal to a right-angled triangle having the two shorter sides equal to the radius of the circle and the circumference of the circle. So the area of the circle

with radius  $OP$  is equal to the area of the triangle  $OPT$ . (3) Note, this is the area of a triangle, and not a square.

Though Archimedes worked hard to solve the first problem, he was unsuccessful in his attempts. A very important fact that should also be mentioned is that if it were not for Conon of Samos, we might not have ever known about the Archimedes spiral. According to Pappus of Alexandria, Conon was a lifelong friend of Archimedes, and Conon was actually the one that found Archimedes' work on the spiral and preserved his work. Conon of Samos was a mathematician and astronomer who worked on conic sections, and contributed to Apollonius' fourth book, 'Conics'.(4)

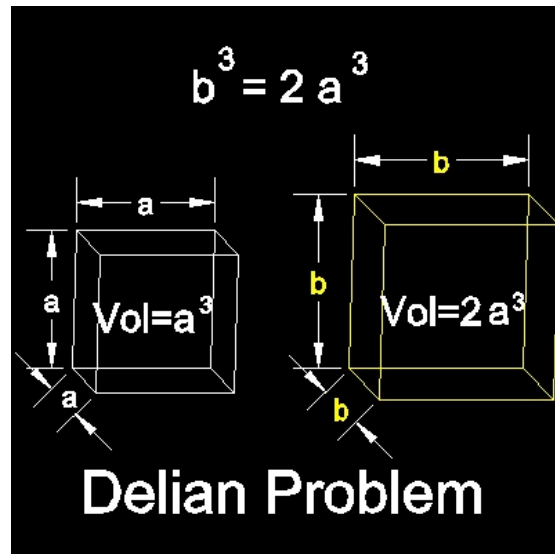
A major step forward in proving that the circle could not be squared using a ruler and compasses occurred in 1761 when Lambert proved that  $\pi$  was irrational.(3) This was not enough to prove the impossibility of squaring the circle with a ruler (measurement), however, it led to a greater interest in the problem.

The final solution to the problem of whether the circle could be squared using the ruler and compass methods finally ended in 1880 when Lindemann proved that  $\pi$  was transcendental. (The definition for a Transcendental is that it is the root of any polynomial equation with rational coefficients). Therefore, the transcendentality of  $\pi$  finally proved that there is not a ruler and compass method for squaring the circle, the problem could not be solved by using this method.

### Problem #2, Doubling the Cube

There are three classical names for the second problem: doubling the cube, duplicating the cube, or the Delian problem. Though the squaring of the circle problem may be the most famous in more modern times, the doubling of the square problem was more popular among the

ancient Greeks. The idea of this problem is to construct the edge of a cube so that it is double the volume of the other cube.



The origination of this problem is quite fascinating, and the story has a hidden meaning.

“Eratosthenes, in his work entitled ‘Platonicus’ relates that, then the God proclaimed to the Delians through the oracle that, in order to get rid of a plague, they should construct an altar double that of the existing one, their craftsmen fell into great perplexity in their efforts to discover how a solid could be made the double of a similar solid; they therefore went to ask Plato about it, and he replied that the oracle meant, not that the God wanted an altar of double size, but that he wished, in setting them the task, to shame the Greeks for their neglect of mathematics and their contempt of geometry.”- Eratosthenes(5)

The plague in Athens was a major event, occurring around 429 BC. This plague appeared three times, and eventually took the life of well known Pericles(6). Historians believe that this story could be the truth to where the second problem originated, since the story did seem to become prominent around 430 BC, and the date is also consistent with an early contribution by Hippocrates.

Hippocrates was one of the first mathematicians to try to solve the Delian problem. His first major step is to find a cube whose ratio to a given cube equals the ratio of two given lines. Then he reduces the first step to show: given two lines, find two mean proportionals between them. With modern understanding, we can see that the ratio's of the first and second are equivalent for:

$$a : x = (a : x) = (a : x)(x : y)(y : b) = a : b.$$

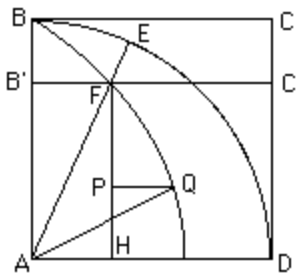
Euclid also showed that the proportions 'C' and 'D' are equivalent in the work of the 'Elements'. Using 'C' to find a square whose ratio to a given square equals the ratio of two given lines; and 'D': given two lines, find one mean proportional between them such that given lines a, b, and x,  $a : x = x : b$ . The modern argument would be  $a : x = (a : x)(x : b) = a : b$ .

Though many great mathematicians tried to solve the second problem, it has been proven impossible. Carl Friedrich Gauss (1777-1855) states without a proof that the problems of trisecting the angle and doubling the cube that it could not be solved. But, when the first proof was published in 1847 by Pierre Wantzel (1814-1848) he showed that these problems were possible.(7)

### Problem #3, Trisecting an Angle

Being the least famous of the three problems, trisecting the angle differs from the first two classical problems. First, there is no particular reason why the problem originated, and second, this problem is the only one that can be solved using a straightedge and compass.

Hippias is usually credited for the major contribution of the quadratrix, which tried to use to solve the squaring the circle and trisecting an angle problems. Pappus used his quadratrix in his fourth book, 'Synagoge', and included an explanation on how to trisect an angle.



“ $ABCD$  is a square and  $BED$  is part of a circle, centre  $A$  radius  $AB$ . As the radius  $AB$  rotates about  $A$  to move to the position  $AD$  then the line  $BC$  moves at the same rate parallel to itself to end at  $AD$ . Then the locus of the point of intersection  $F$  of the rotating radius  $AB$  and the moving line  $BC$  is the quadratrix. Hence

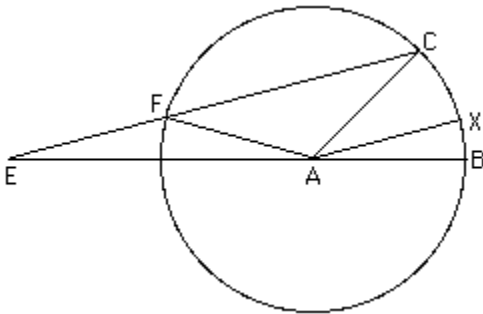
$$\text{angle } EAD / \text{angle } BAD = \text{arc } ED / \text{arc } BED = FH / AB,$$

so, taking  $AB = 1$ ,

$$\text{angle } EAD = \text{arc } ED = FH \times \pi/2.$$

To divide the angle  $FAD$  in a given ratio, say  $p : q$ , then draw a point  $P$  on the line  $FH$  dividing it in the ratio  $p : q$ . Draw a line through  $P$  parallel to  $AD$  to meet the quadratrix at  $Q$ . Then  $AQ$  divides angle  $FAD$  in the ratio  $p : q$  (7).

There is also another mechanical solution given by Archimedes. This is from his ‘Book of Lemmas’:



Given an angle  $CAB$  draw a circle with center  $A$  so that  $AC$  and  $AB$  are radii of the circle. From  $C$  draw a line to cut  $BA$  produced at  $E$ . Have this line cut the circle at  $F$  and have the property that  $EF$  is equal to the radius of the circle.

Again this can be done in a mechanical way by marking a length equal to the radius of the circle on the ruler and moving it keeping one mark on  $BA$  produced and having the second mark on the circle. Move the ruler keeping one mark on the

line and the other on the circle until the ruler passes through  $C$ . Then the line  $EC$  is constructed. Finally draw from  $A$  the radius  $AX$  of the circle with  $AX$  parallel to  $EC$ . Then  $AX$  trisects angle  $CAB$ .

So, Angle  $XAC = \text{angle } ACF = \text{angle } CFA = \text{angle } FEA + \text{angle } FAE = 2 \times \text{angle } FEA = 2 \times \text{angle } XAB$ . (7)

### Conclusion of the Three Famous Problems

The three problems, even though the first two are not possible, have led to many great discoveries and advancements in mathematics. It has led to the discovery of conic sections, cubic

and quartic curves, and several transcendental curves. There are many solutions to solve the Trisection problem, however the first two still amaze some great minds of our time.

### Zeno's Paradoxes

Almost everything we know about Zeno is found in the opening pages of Plato's 'Parmenides'. His birth is estimated around 490 BC, and he apparently wrote about 40 'paradoxes of plurality.'

A paradox is a demonstration that a contradiction or absurd consequence follows from apparently reasonable assumptions. (8)

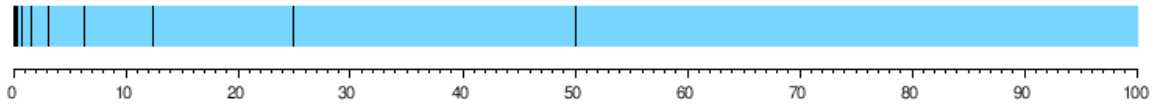
According to Aristotle, Zeno was the inventor of the method of Dialectic. It is the technique for arguing for or against a position by careful logical reasoning. He used this method to defend his friend Parmenides by attacking his (Parmenides') critics. Parmenides rejected pluralism and the reality of any kind of change. For him, all was one indivisible, unchanging reality, and any appearances to the contrary were illusions, to be dispelled by reason and revelation. Zeno attempted to show that equal absurdities logically followed Parmenides' views.

Zeno's four main paradoxes are (9):

#### **A) The Racecourse, or dichotomy paradox.**

Suppose a runner needs to travel from a start S to a finish F. To do this he must first travel to the midpoint, M, and thence to F. But, if N is the midpoint of SM, he must first travel to N, and so on *ad infinitum*. Therefore, the runner cannot complete (or start) his journey because he will always have to travel half the distance of every

segment before completing his journey.



$$H - \frac{B}{8} - \frac{B}{4} - \frac{B}{2} - \dots - B \left\{ \dots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1 \right\}$$

**B) Achilles and the Tortoise.**

Achilles runs a race with a tortoise, in which the tortoise has a start of  $n$  meters.

Suppose the tortoise runs a tenth as fast as Achilles. Then by the time Achilles has reached the tortoise's starting point, the tortoise is  $\frac{n}{10}$  meters ahead. By the time

Achilles has reached that position, the tortoise is  $\frac{n}{100}$  meters ahead, and so on *ad*

*infinitum*. So Achilles cannot catch the tortoise.

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

**C) The Arrow.**

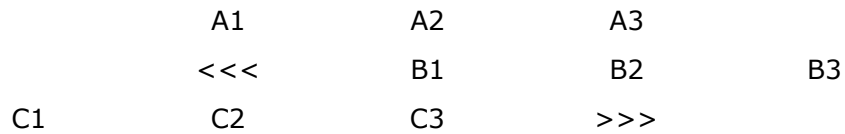
An arrow cannot move at a place at which it is not. But, neither can it move at a place at which it is. But a flying arrow is always at the place at which it is. That is, at any instant it is at rest. But if at no instant is it moving, then it is always at rest. (9)

Therefore: in any instant, the arrow is in one place, and one place only. So the fact that it is moving may be just an illusion!

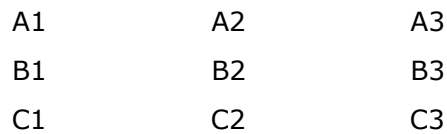
**D) The Moving Blocks, or Stadium.**

Suppose three equal blocks A, B, C, of width I, with A and C moving past B at the same speed in opposite directions. Then A takes one time, t, to traverse the width of B, but half the time,  $\frac{t}{2}$  to traverse the distance I. (9) If this takes place, then they will be in on the same level at the same time.

**Position 1**



**Position 2**



Two other popular paradoxes proposed by Zeno are:

A) The Paradox of Place: If everything that exists has a place, place too will have a place, and so on *ad infinitum*. This tends to be a cycle because using the word “place” implies that place itself will also need a “place” to belong.

B) The Grain of Millet: even though his reasoning is false, he argues that there is no part of the millet that does not make a sound. For there is no reason why any part should not in any length of time fail to move the air that the whole bushel moves in falling. (I believe he is implying that with the movement there is sound, and since everything is moving in the bushel, everything is making a sound at that moment.)

Zeno's arguments are perhaps the first examples of a method of proof called *reduction ad absurdum*, also known as a proof by contradiction.<sup>(9)</sup> Mathematicians claim to have done away with Zeno's paradoxes with rigorous analysis of the units of distance and time involved in the problem, and with the invention of the calculus methods of handling infinite sequences. However, some philosophers insist that the deeper metaphysical questions, as raised by Zeno's paradoxes are not addressed by the calculus.

In 1977, physicists E.C.G. Sudarshan and B. Misra who were studying quantum mechanics discovered that the dynamical evolution (motion) of a quantum system can be hindered through observation of the system. This effect is usually called the quantum Zeno effect as it is strongly reminiscent of the Zeno's arrow paradox.

### General Conclusion

The end of the Hellenic Period was marked by the death of Alexander the Great in 323 BC. Though his empire remained, it was divided. Aristotle's works and his great mind were looked down upon by the new ruling, so he left Athens soon after. A year after leaving Aristotle died.

The Hellenistic Era in history was an period of great thinking and discovery not only in the mathematical but astronomical and philosophical worlds. Events that took place not only gave rise and advancements to the mathematics we use today, but problems proposed by the ancient Greeks are still being studied and proved in modern times.

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