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Abstract

Leonardo da Pisa or Fibonacci was a man who made a great influence on the mathematics we study today. He learned about Arabic numerals while traveling with his father and helped to introduce them to Europe. He found the sequence of numbers we know today as the Fibonacci sequence, and discovered many properties regarding it and square numbers in his books. Even with all of his accomplishments he greatly under appreciated and does not often receive credit for many things he discovered and worked on.

The Life and Works of Leonardo da Pisa

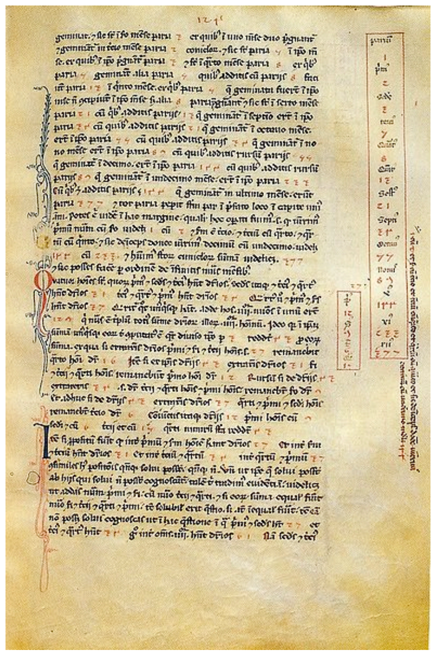
To most people the name of Leonardo da Pisa does not have significant meaning. He was just a man who lived then died. What did he contribute to the world? Then when the name Fibonacci is said it is recognized by many more people. “Fibonacci made great advances in mathematics,” someone might say, or “he has that sequence of numbers I learned about in school.” So what do these two men have in common with each other? Leonardo da Pisa and Fibonacci are not two men, but one man. While it is true there is a sequence of numbers that bears his name, it is far from the only contribution he made to mathematics.

Possible sketch of Leonardo da Pisa[1]

The mathematician the world refers to as Fibonacci was born as Leonardo da Pisa in the year 1170. He was most likely born in Pisa, now in modern day Italy. Little is known about his family and personal life outside of his achievements in mathematics. It is known that his mother, Alessandra, died when he was young. His father Guglielmo, who was addressed by the name Bonaccio, worked in North Africa as a diplomat. After his father's death Leonardo was given the name Fibonacci, meaning son of Bonaccio, in remembrance of his father. While Guglielmo worked in North Africa, Fibonacci was educated in Egypt, Syria, and Greece. There Fibonacci learned the Arabic numeral system.

Fibonacci returned to Pisa in the year 1200. Upon returning he wrote several books. Of these books only copies of *Liber abaci* (*The Book of the Abacus* or *The Book of Calculation)*, *Practica geometriae* (The Practice of Geometry), *Flos*, and *Liber quadratorum* (The Book of Squares) have survived. Historians know some of the works of Fibonacci have been lost, due to the limited ability to recreate works at the time. A couple of these lost works are *Di minor guisa,* and his commentary on Book X of Euclid's *Elements.*

The most popular work of Fibonacci is *Liber abaci* (*The Book of the Abacus* or *The Book of Calculation). Liber abaci* was published in 1202, and is an account of all the arithmetic and algebra that Fibonacci learned during his travels.



Page from Liber abaci

One of the major milestones of the Middle Ages was the shift from the use of Roman Numerals to the use of Arabic Numerals. The first section of *Liber abaci* introduced Arabic numerals which contained place-valued decimals. *Liber abbaci* begins

> *"These are the nine figures of the Indians: 9 8 7 6 5 4 3 2 1. With these nine figures, and with the sign 0 which in Arabic is called zephirum, any number can be written as will be demonstrated"*

This statement begins the explanation of the Arabic rules for working with the Indian Numerals. Fibonacci wrote mixed fractions by placing the fraction to the left of the integers, and also used a fraction bar. While this was not the first appearance of these numerals in Europe, it is believed to be the one of the most influential.

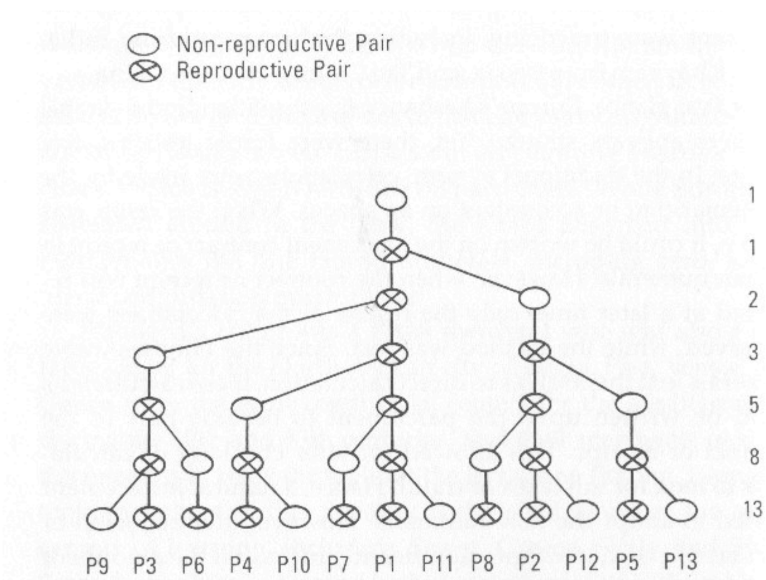
The second section of *Liber abbaci* contains a set of problems aimed towards readers that were merchants. The information contains the algebra used to compute price, profit, and currency exchange with Arabic numerals. It took Europe a significant amount of time to begin to use this number system. The first people to use Arabic numerals were merchants. The use of Roman numerals was difficult for merchants. All calculations needed to be done on a counting board or on an abacus and not everyone knew how to use an abacus. After prices were calculated the final result was written on a slip of paper as a receipt. Because the amount of people who knew how to use an abacus was limited most customers were not able to check how accurate calculations were. The use of Arabic numerals changed this. Since the numerals themselves lend in calculations the merchants could learn to calculate prices and exchange rates. They also could write a proper receipt for customers. Customers could then check calculations and spot errors, making fraud easier to spot (Gies 28) (Ball 168).

Fibonacci also discussed "common finger computations" and the use of Roman numerals in this section. As stated Roman numerals were still the most common computational method in Europe at the time the book was written. While Fibonacci said one of the reasons he wrote *Liber abbaci was*, "in order that the Latin race might no longer be deficient in [the] knowledge [of Arabic numerals]" he also knew Roman numerals were the most well known (Clawson 123).

The third section of the book has the information that the world knows today as the Fibonacci sequence. By using the question

> *"How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on?"* (Clawson 125).

The following picture helps depict how the sequence is found from this question.

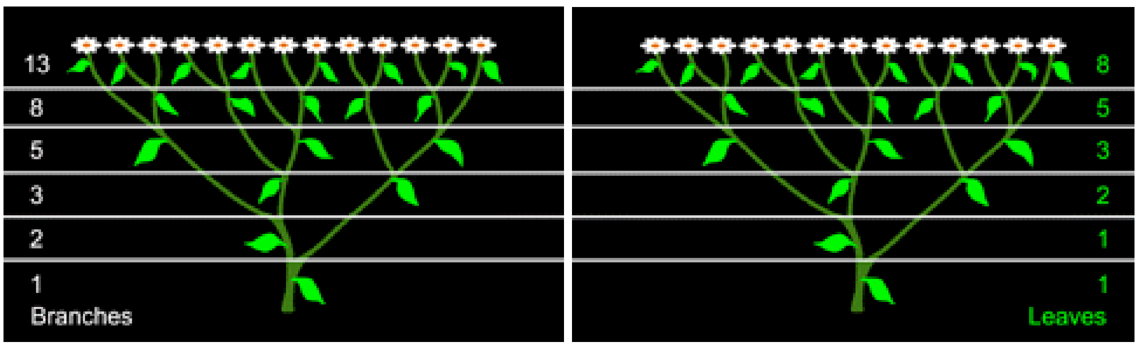


We discover a sequence of numbers in which one term is the sum of the pervious two terms. The sequence begins as follows: 1,1,2,3,5,8,13,21,34,55,89,144,233… and continues infinitely.

Fibonacci's sequence has several unique properties in nature and in mathematics. In nature the sequence seems to be a blueprint for growth and generation of living organisms. Many flowers have an amount of petals that is a piece of the Fibonacci sequence. The calla lily has only one petal and the euphorbia has two petals both of which are very rare. Daisies are also known to have 13,21,34,55, or 89 petals.

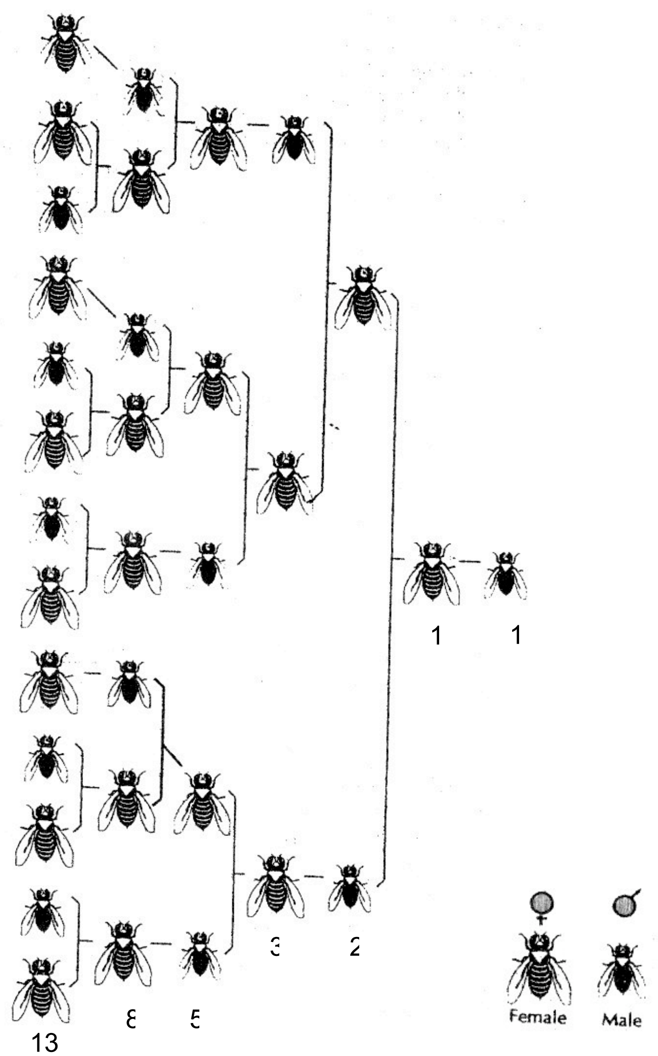
(From Fibonacci Numbers in Nature)

The sequence does not only appear in the petals of flowers. It can also appear in the branches and leaves, such as the sneezewort.



(From Fibonacci Numbers in Nature)

Many other similar examples exist. There is also another example in nature on how to find the Fibonacci sequence, by looking at the genealogy of the male bee. Each male only has one parent, a mother. Female bees have two parents. So after we look at the following picture we see that a Fibonacci sequence is created.



In addition to all of its appearances in nature the Fibonacci sequence has many mathematical properties. For example any two consecutive Fibonacci numbers (F_n, F_{n+1}) will have the property gcd(F_n, F_{n+1})=1. In other words they will be relatively prime to one another. The sum of the first n even Fibonacci terms is F_{2n+1}-1.

Proof by induction:

Base Case:

$$n = 1$$
$$F_{2n} = F_{2n+1} - 1$$
$$F_2 = F_3 - 1$$
$$1 = 2 - 1$$
$$1 = 1$$

Assume:

$$F_2 + F_4 + \ldots + F_{2k} = F_{2k+1} - 1$$

WTS:

$$F_2 + F_4 + \ldots + F_{2k} + F_{2k+1} = F_{2(k+1)+1} - 1$$

$$F_2 + F_4 + \ldots + F_{2k} + F_{2k+1}$$
$$= F_{2(k+1)+1} - 1 + F_{2k+1}$$

Because of the properties of Fibonacci number, when you add two consecutive Fibonacci numbers you get the next Fibonacci number

$$= F_{2k+1} + F_{2(k+1)+1} - 1$$
$$= F_{2k+3} - 1$$
$$= F_{2(k+1)+1} - 1$$

QED

Another similar property of the Fibonacci numbers is the sum of the first n odd Fibonacci numbers is given by F_{2n}.

Proof by induction:

Base Case:

$$\begin{aligned} n &= 1 \\ F_{2n-1} &= F_{2n} \\ F_{2(1)-1} &= F_{2(1)} \\ F_1 &= F_2 \\ 1 &= 1 \end{aligned}$$

Assume:

$$F_1 + F_3 + \ldots + F_{2k-1} = F_{2k}$$

WTS:

$$F_1 + F_3 + \ldots + F_{2k-1} + F_{2(k+1)-1} = F_{2(k+1)}$$

$$F_1 + F_3 + \ldots + F_{2k-1} + F_{2(k+1)-1}$$
$$= F_{2k} + F_{2k+1}$$

Because of the properties of Fibonacci number, when you add two consecutive Fibonacci numbers you get the next Fibonacci number

$$= F_{2k+2}$$
$$= F_{2(k+1)}$$

QED

Many other properties for the Fibonacci sequence are given in the third section. There are also problems involving perfect numbers, and the Chinese Remainder Theorem.

The popularity of *Liber abaci* grew over time. Many across Europe began to hear about Fibonacci and about his skills. In 1225 Frederick the II came to Pisa to hold a mathematical tournament. He had heard of Fibonacci and wanted to test his skills. The other competitors failed to solve any questions even though they were given the questions ahead of time while Fibonacci was able to answer all of them (Ball 169).

Another book by Fibonacci, *Practica geometriae* was written in 1220. This book contains eight sections with geometry based on Euclid's *Elements* and Euclid's *On Divisions*. Even though the book contains precise proofs, it also includes how to solve problems for practical situations using similar triangles. The final chapter contains what Fibonacci called geometrical subtleties.

> *"Among those included is the calculation of the sides of the pentagon and the decagon from the diameter of circumscribed and inscribed circles; the inverse calculation is also given, as well as that of the sides from the surfaces. ... to complete the section on equilateral triangles, a rectangle and a square are inscribed in such a triangle and their sides are algebraically calculated ..."*(Fibonacci biography).

Liber quadratorum, written in 1225, is thought by some to be Fibonacci's most remarkable piece of work. Specifically this book contains 24 propositions about the properties of square numbers and finding them. Additionally the book contains number theory including, most significantly how to find Pythagorean triples. To further examine Pythagorean triples the way Fibonacci did, we first need to know about square numbers. Fibonacci said,

> *"I thought about the origin of all square numbers and discovered that they arose from the regular ascent of odd numbers. For unity is a square and from it is produced the first square, namely 1; adding 3 to this makes the*

> *second square, namely 4, whose root is 2; if to this sum is added a third odd number, namely 5, the third square will be produced, namely 9, whose root is 3; and so the sequence and series of square numbers always rise through the regular addition of odd numbers."*(Book of Squares)

Which in simpler terms means, "that square numbers can be constructed as sums of odd numbers, essentially describing an inductive construction using the formula $n^2 + (2n+1) = (n+1)^2$". Or written in equation form is

1 = 1_

1 + 3 = 2_

1 + 3 + 5 = 3_

1 + 3 + 5 + 7 = 4_

and suggests the general formula, which we would express as:

1 + 3 + ... + (2n-1) = n_

Proof by induction:

Base Case:

$$n = 1$$
$$2n - 1 = n^2$$
$$2(1) - 1 = (1)^2$$
$$1 = 1$$

Assume:

$$1 + 3 + ... + (2k - 1) = k^2$$

WTS:

$$1 + 3 + ... + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2$$

$$\begin{aligned} &1 + 3 + ... + (2k - 1) + (2(k + 1) - 1) \\ &= k^2 + (2(k + 1) - 1) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

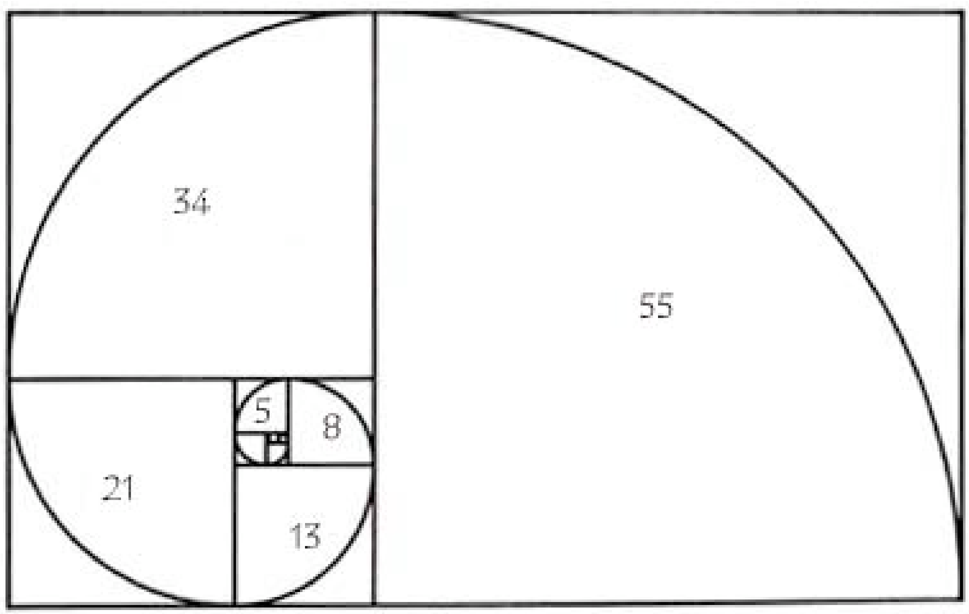
QED

Once we understand square numbers we can build Pythagorean triples.

> *"Thus when I wish to find two square numbers whose addition produces a square number, I take any odd square number as one of the two square numbers and I find the other square number by the addition of all the odd numbers from unity up to but excluding the odd square number. For example, I take 9 as one of the two squares mentioned; the remaining square will be obtained by the addition of all the odd numbers below 9,*

namely 1, 3, 5, 7, whose sum is 16, a square number, which when added to 9 gives 25, a square number."(Book of Squares)

Along with Fibonacci's sequence there is something called Fibonacci's spiral. "A Fibonacci spiral approximates the golden spiral; unlike the "whirling rectangle diagram" based on the golden ratio, this one uses squares of integer Fibonacci-number sizes, shown for square sizes 1, 1, 2, 3, 5, 8, 13, 21, and 34."(Fibonacci)



The ratio between any two Fibonacci numbers after 3 is about 1 to 1.6. This is approximately the golden ratio. This ratio is said to be one of the most visually satisfying of all geometric forms (Bergamini 94). This ratio has been seen in buildings as old as the temples of Athens too the paintings of Leonardo da Vinci.

Fibonacci's work in number theory was almost totally ignored by the mathematical community during the Middle Ages. This was despite that fact he had a reputation with kings and emperors. Three hundred years after Fibonacci we find the same ideas about number theory appearing in the work of Maurolico. Some of which Maurolico claimed was his own.

Fibonacci is believed to have died in 1250. The exact day and even the year is uncertain. 1250 is the best estimate from historians considering it was the last time any work from Fibonacci was seen.

Leonardo da Pisa was given many challenges throughout his life. His mother died when he was young and he was raised by a father who was constantly traveling. It was partly because of this traveling Leonardo became the person we know today as Fibonacci. He did many great things for mathematics by discovering his sequence and its unique properties. He also found several properties of square numbers. None of his accomplishments may be greater than the one he is given the least credit for, which is bringing Arabic numerals to Europe. Because of this one event we are able to perform calculation easier than ever before. The Arabic numerals also gave more quantities of people the chance to be able to calculate numbers. His goal was to spread knowledge and he did this not just by showing the kings and the wealthy but by presenting a new method to broaden possibilities.

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