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MTH 467

Euclid

Abstract: In this paper we will discuss the mathematician Euclid. We will talk about Euclid's history and his greatest work, *The Elements*.

Introduction:

Euclid, pronounced "YOO-klid," is an Ancient Greek name derived from the words for "good and glory." (Campbell) Any student of mathematics, from high school geometry to college Euclidean Geometry, is sure to become familiar with this name. Euclid certainly lived up to his name in his works and achievements. We will examine Euclid's life and one of his greatest achievements, *The Elements*.

Euclid's History:

Looking into recorded history, the further back in time you go the vaguer the details become. So much has been lost as the years passed, so of course there is some argument about Euclid and his life. Many confuse Euclid of Alexandria with Euclid of Megara. Euclid of Megara lived approximately 100 years before Euclid of Alexandria. Euclid of Megara was a Greek philosopher who studied under Socrates in Athens. Some say that Euclid was merely a name adopted by a group of mathematicians working at Alexandria who admired the Greek philosopher. Others believe that Euclid was a leader of a group of mathematicians at Alexandria and a historical character who wrote *The Elements*. Each of these views is arguable, though most evidence points to Euclid as a man who wrote *The Elements* as a teaching tool for the school at Alexandria. Proclus, a Greek philosopher (410-485 CE), wrote:

> "Euclid put together the Elements... This man lived in the time of the first Ptolemy. For Archimedes, who came immediately after the first, makes mention of Euclid: and, further, they say that Ptolemy once asked him if there was in geometry any shorter way than that of the elements, and he answered that there was no royal road to geometry. He is then younger than the pupils of Plato but older than Eratosthenes and Archimedes; for the latter were contemporary with one another, as Eratosthenes somewhere says." (Heath)

This statement gives us most of what is known about Euclid and places him in Alexandria about 300 BCE. Euclid is featured in paintings by Raphael and other artists as a great teacher and geometer. *A History of Mathematics* by Carl Boyer also pictures Euclid as a great teacher. His works, including *The Elements* and the others that were lost, all seem to have been written to be used as teaching tools. *The Elements*, in fact, was the leading mathematics textbook for 2000 years. It was also one of the first mathematics textbooks printed on the printing press, and has over 1000 editions. Only the Bible exceeds it in the number sold.

Euclid's *Elements*:

Euclid wrote at least 10 works, all but the *Elements* were lost in time. We know of these works as they are referred to by others. *Data* was a book that is said to have contained 95 exercises for students who had completed the *Elements*, painting Euclid yet again as the wonderful teacher. The most grieved loss is *Porisms*. In Greek, porism is often used to mean corollary. It is thought that this was a book of advanced Geometry meant to follow up and build upon the *Elements*.

The *Elements* Acclaim:

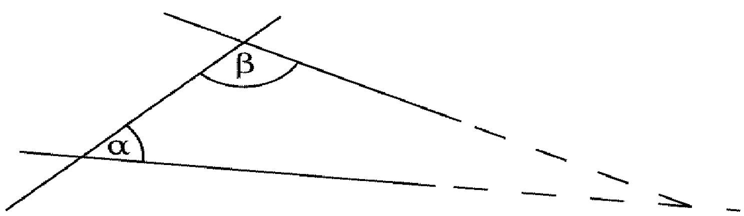
The *Elements* are held in very high esteem. "Euclid's *Elements* form one of the most beautiful and influential works of science in the history of humankind… It has influenced all branches of science but none so much as mathematics and the exact sciences." (Joyce, 1997) Looking at mathematics textbooks today, could we say that any of them are beautiful? Is it the language that Euclid uses that inspires us to say the *Elements* is attractive? I believe it is the compilation of so much mathematics that makes it lovely. "Euclid speaks to us with a voice as clear and universal as laughter. The simplicity, clarity and elegance of Euclid's proofs… both delight and instruct." (Heath, 1926) How many great teachers today can we say "delight and instruct" us? This statement is very bold, and shows how great of a teacher Euclid really was. A person's character shines through in their work. Euclid's character shines as brightly as his delightful work, the *Elements*. Today, mathematics text books range in cost from ten to hundreds of dollars, and none shine as brightly to the modern student as the *Elements* has shown.

All other books lie in the shadow that the *Elements* have cast on them. None has gained such acclaim; it does not seem possible that they could.

How did the *Elements* come to be so grand? Why is it not merely another mathematics text book? Let's take an in-depth look at how the *Elements* was put together. Euclid took advantage of his location in Alexandria and his education to write the *Elements*. He has been likened with Webster in that he compiled information into a book that could be utilized by all. This comparison is slightly unfair because not only did he compile information, but he added to it and expounded upon it. Much of the more "obvious" proofs were left up to the reader to prove, but Euclid included proofs to the difficult theorems. In total, the *Elements* contains 126 definitions and 468 propositions. Though Euclid is referred to as the great geometer, the *Elements* is not merely a book on geometry. It is written in what I would call a geometric language, but you will find the foundations of algebra among the lines and figures. A mere cursory glance at the *Elements* would mislead you and dishonor the great text. As the books of the *Elements* are unfolded, they build upon previous definitions and postulates, thereby becoming more involved and abstract. As a well constructed book should be, Euclid completes needed information before moving on to the more complicated. He doesn't refer to theorems later in the book as proof for propositions that appear earlier, unlike many modern textbooks.

Book I:

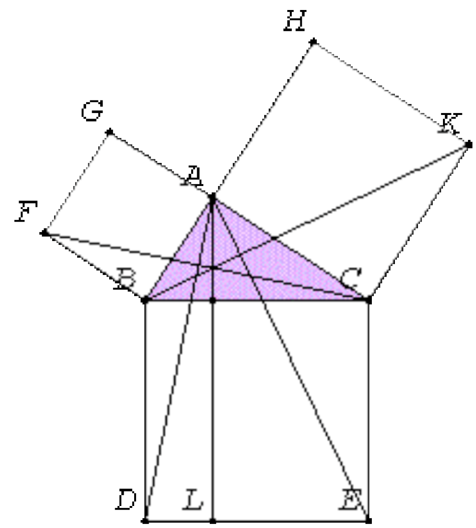
Book I of the Elements sets up the foundations starting with fundamentals of geometry such as theories of triangles, parallel lines, and area. The first definition is of a point, "which has no part.” (Heath) It goes on to define lines, angles, boundaries, circles, triangles, and much more. Notable is his definition of parallel lines. “Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.” (Heath) Euclid’s parallel postulate is the 5th listed in the Elements and states “if a straight line falling on two straight lines make the interior angle on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.”(Heath, 1926) That is, if two lines, lying in the same plane, intersect the same line and form acute interior angles _ and _, then producing these lines indefinitely you will find that they intersect on the side of the acute interior angles.



(Wikipedia, 2008)

If _ and _ are supplementary angles, then the lines are parallel.

In this book, we also find a proof of the ever famous Pythagorean Theorem. This construction is referred to as the pinwheel, because the figure resembles a pinwheel. The proof that in right-angle triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle is as follows:



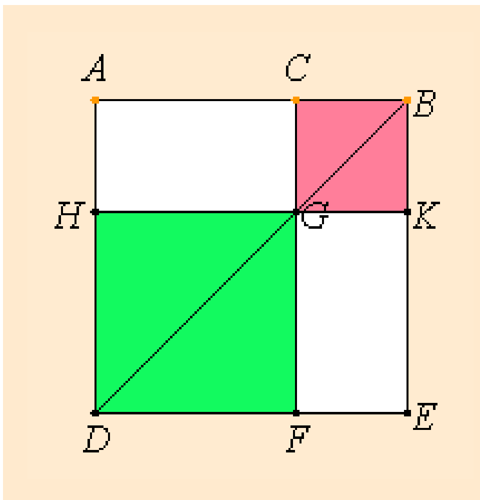
(Joyce)

Let ABC be a right triangle. Draw squares using each side of the triangle as shown in the figure above. Also, draw line AL parallel to BD such that DLE. We will show that square FGAB is equal to parallelogram BDL. Connecting points F and C, we get triangle FBC. Also, connecting points A and D we get triangle ABD. Angle FBC is equal to angle ABD since angle FBA equals angle CBD because they are both right angles and angle ABC added to the right angle gives angle FBC and angle ABD. So triangle FBC is equivalent to triangle ABD by side-angle-side. We also have that the area of square FGAB equals twice the area of triangle FBC since they have base FB and height AB. For this same reason, the area of parallelogram BDL is twice the area of triangle ABD. It follows that the area of FGAB is equal to the area of parallelogram BDL. The proof is similar for the area of AHKC equal to the area of parallelogram CEL. Therefore, BCED is equal to FGAB added to AHKC. The theorem is proved.

It is in this proof that we can clearly see how geometry is used to represent a known algebraic equation for a right triangle, $a^2+b^2=c^2$. Our favorite values, a, b, and c are represented in the line segments on the triangle. For squared magnitudes, we see squares represented on the figure. Using properties of parallel lines, triangles, and parallelograms, we find that the two smaller squares built from the smaller sides do in fact add up to the larger square built on the hypotenuse.

Book II:

Similar geometric representations of algebraic properties that are taught in grade school to be absolute truths can be found throughout Euclid's *Elements*. In fact, Book II is referred to as Geometric Algebra. Here we find a proof of $(x+y)^2=x^2+2xy+y^2$.

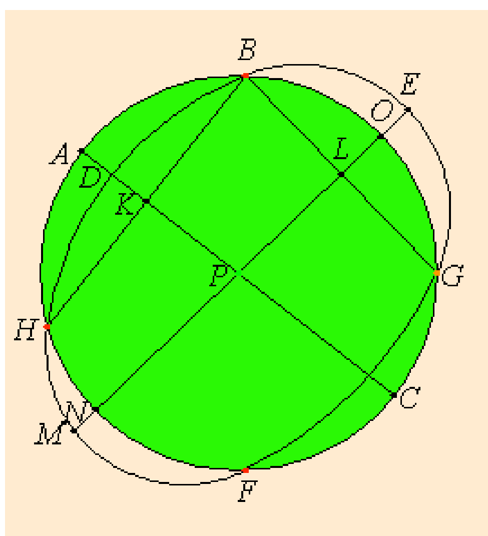


(Joyce)

Given this figure, an algebra student is much more likely to understand the true concept behind the well know algebraic property. In this case, let x be the line segment HG and y be the line segment GK. We know that HG added to GK is HK. Since HK is equal to DE and AD, we can see that DEBA is HK squared, this is our $(x+y)^2$. HGFD is x^2 and GKBC is y^2. The areas of two white rectangles, ACGH and GKEF, are equal to x times y, which is where we find our 2xy. Adding the areas of the inner figures, the student will find that the property is evident.

Book III:

Book III expounds on circles and angles within the circle. Let's look at a proof of one of the propositions Euclid presents on circles. We will show that a circle does not intersect another distinct circle at more than two points. By way of contradiction, assume that the circles intersect at more than two points. Given circles ABC and DEF, let them intersect at points F, H, B, G as shown in the figure.

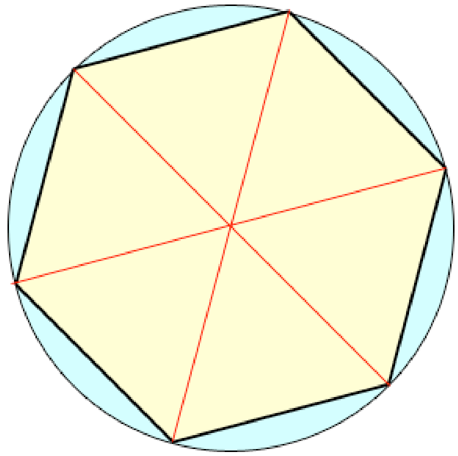


(Joyce, 1997)

Draw segments BH and BG. Let BH be bisected at point K by segment AC such that AC is perpendicular to HB. Let BG be bisected at point L by segment LM such that LM is perpendicular to BG. Since in circle ABC, AC bisects BH at right angles, the center of ABC is on the line AC. For this same reason, the center of ABC is on the line LM. LM and AC intersect only at point P, therefore P is the center of ABC. Similarly, we can show that P is the center of DEF. But then, ABC and DEF would have the same center P. This is a contradiction. Therefore, ABC and DEF cannot intersect at more than two points. The proposition is proved. In this book we also find Thales theorem of right, acute, and obtuse angles found in a semicircle. Let ABC be a triangle inscribed in a circle. If one side of the triangle is equal to the diameter of the circle, then it is a right triangle.

Book IV:

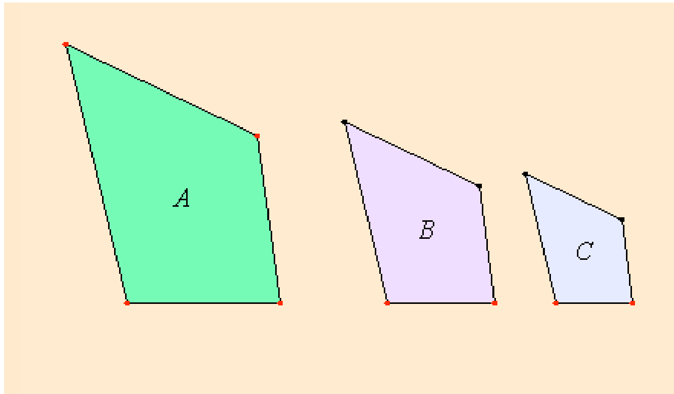
Book IV shows how to build regular polygons based on a specific triangle, the isosceles triangle. Any regular polygon can be broken down into isosceles triangles. Let's look at a square. Drawing in a diagonal line connecting one vertex to the only vertex that is not on the same line, we divide the square into two parts. Since it is a square and the sides are all equal, we have two isosceles triangles. Drawing line segments from the center of the square to each vertex, you will find 4 isosceles triangles. This same construction can be repeated for any regular polygon.



(Joyce, 1997)

Book V and Book VI:

In Book V we move into abstract algebra, ratio, and proportion. Euclid compares different magnitudes represented by line segments. He states and proves many truths that are evident to the modern mathematics student. We must keep in mind that the properties we often take for granted today had to be built from somewhere. Something as simple as: if two numbers, say A and B have the same ratio to a number C, then A and B are equal. We might see it in fraction form: A= _ C and B= _ C. It is clear that A and B must be the same number. Euclid extends this property into many more parts, proving that it is true, even if you have more than 2 magnitudes equal to the same ratio. In Book VI, Euclid takes these properties and applies them to figures. He makes statements and proves such facts as "figures which are similar to the same figure are also similar to one another." It is the same logic, only with more dimensions.



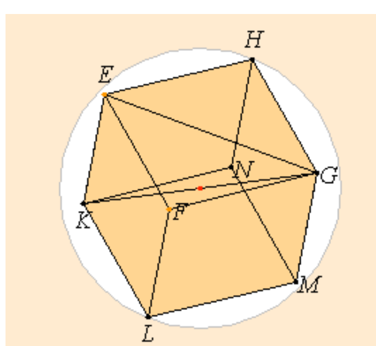
(Joyce, 1997)

Books VII, VIII and IX:

Books VII, VIII and IX move into number theory. In these sections of the book we find prime numbers, the Euclidean algorithm, and multiplication. The Euclidean algorithm is used to find the greatest common divisor of two magnitudes. Using repeated subtraction, also known as division, Euclid isolates the largest number that will divide two distinct numbers. Book VIII applies proportions to number theory. In Book IX we find the proof that there are infinitely many primes. Let A, B, and C be prime numbers. There is at least one more prime number than these. Let A*B*C=D. Let's add E to D to get a new number F. F is either prime or not prime. If F is prime, then we have found one more prime number. If F is not prime, then it is divisible by a prime number, say G such that G does not equal A, B, or C. Since A, B, and C divide D, then G also divides D. But, G divides F. This is a contradiction because then G would divide E. So, we have found one more prime, G. Therefore, there are infinitely many primes.

Books X, XI, XII, XIII:

Many of the proofs in this section can be a little confusing because the numbers are represented with line segments. The calculations are more in-depth, so it is harder to wrap your mind around. The propositions and proofs become longer and more involved. There are nine previous books that Euclid is building on. In fact, Books I-IX only comprises half of the entire work! In Book X, Euclid introduces commensurable and incommensurable magnitudes. "Those magnitudes are said to be *commensurable* which are measured by the same measure, and those *incommensurable* which cannot have any common measure." (Joyce, 1997) What this means is: two numbers A and B are commensurable if there is another number C such that A and B are multiples of C. That is, there are numbers m and n such that $nC = A$ and $mC = B$. Books XI, XII, and XIII take all of the properties previously established and apply them in more dimensions. Thus far in the *Elements* Euclid has proved theorems concerning magnitudes, or lines, then two dimensional figures. The next logical step is solid figures, or three dimensional figures. Euclid gives construction and properties of these solid figures in the half of the book. It is funny that for objects that take up space, these few books take up more space than the others. There are more things to consider as you move mathematics into more dimensions.



(Joyce, 1997)

Conclusion:

"Good and glory" (Campbell, 2009) certainly do describe Euclid of Alexandria well. His works were both great and glorious. Though we know little about the man himself, his achievements in the *Elements* have shone brightly across the centuries, and are sure to shine for many years more. Euclid was arguably one of the greatest mathematics teachers that ever lived, each student of geometry adds a little more to his great legacy.

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