

A Brief Look at the relationship between MUSIC AND MATHEMATICS

R. Charmaine Hedstrom
Math 470

1 Introduction

Music is something that is simply enjoyed by everybody, but the essence of which is something that is generally obscure from the naked eye. The more one studies music the more one will discover its hidden mathematical treasures. One of the keys to being a good musician is to understand the mathematics, which unlocks the secrets of music.

Sound in itself is a science with various vibrations per second, and terms such as amplitude, decibels, hertz, duration, intensity and timbre, are all measured by numbers. String vibrations cause various sounds called partials or overtones depending on the number of waves created by a single length. The tuning of instruments uses the number of sound pulses per second to determine notes.

2 Basic Music Notation and its Corresponding Numeric Values

2.1 Counting

The general beat of music can be broken down into various fractions. Figure 2.1 shows typical amounts of beats per note, assuming we are using the quarter note as a beat which has a value of one. These notes and their corresponding rests are the five most frequently used in music.

The notes are arranged on a musical score in predetermined groupings of any amount such as 4-beats, 3-beats, 2-beats, etc as the composer desires. Each group is called a measure and is separated by a line called a “bar line”.

Using the table above and corresponding note values, observe in the sample below that the notes in each measure add up to the top number which is at the beginning of each line. The other number, which is at the bottom refers to the quarter note. Thus in the first line there should be the equivalent of 2 quarter notes per measure. The second line should have 3 quarter notes per measure and likewise in the third row each measure must have the equivalent of 4 quarter notes.

Figure 1: Lengths of notes




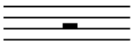



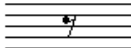

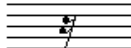
Name	Note	Rest	
Whole Note			Whole note: 4 beats
Half Note			Half note: 2 beats
Quarter Note			Quarter Note: 1 beat
Eighth Note			Eighth Note: 2 notes per beat
Sixteenth Note			Sixteenth Note: 4 notes per beat or

Figure 2.1 illustrates this.

3 Keyboard

If you were to look at a piano keyboard you will notice black keys and white keys. The black notes are in groups of twos and threes. Notice how the white keys are named on the diagram below:

Figure 3 shows how the white keys are named on the diagram below:

[1]

The white notes are named A,B,C,D,E,F,G. That is 7 notes at a time, which keep repeating. Thus together with the black notes we have a total of 12 notes recurring. It is with these 12 notes that there are an almost countless array of combinations and permutations. The space between two of the same named keys such as C to C is called an octave.

Music is written in various "keys". Simply stated, a tune only uses 7 out of the 12 notes at a time. The seven notes are chosen as follows: For the key of C, that is starting on C play only the white notes. You will notice that there is for example a black note between C and D, but no note between E and F. We call the step from E to F a "half-step" and the step which has a note in between, for instance from C to D, we say this is a whole step. Thus, in the music line below we observe the general pattern as follows, starting at C: Whole-step, Whole-step, Half-step, Whole-step, Whole-step, Whole-step, Half-step.

If we started the pattern on any other key, say for example, the key of E. The second note in the sequence would be the black note to the right of F namely F-sharp or F#. The second whole step from this position would be G-sharp or G# the next half-step would be A, and so on. For simplicity sake, the explanations from this point on will only be using the key of C, or C-scale to illustrate points since everything that applies to this key applies to all 12 keys.

The particular C that is illustrated at the beginning of this line is called "middle C" since it represents the C in the middle of the piano keyboard. The last note of the line is also a "C", but it is the next C up from middle C.

Figure 3 describes the C Major scale.

Whole-step-Whole-step-Half-step-Whole-step-Whole-step-Whole-step-Half-step [3]

The scales that have just been described are known as "major scales". Below is the general description of the "minor scales". Again, we use C-minor as a typical example of the pattern. Notice how the order of the whole-steps and half-steps change. The third, sixth and seventh keys are played a half-step lower.

Figure 3 describes the C Minor scale.

Figure 2: Beats per Measure



Figure 3: Keyboard



Figure 4: C Major Scale

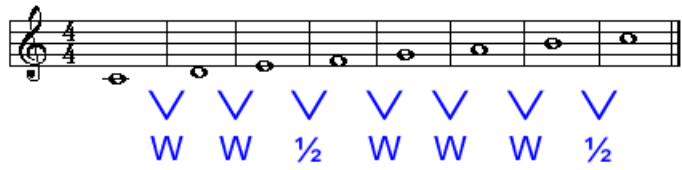
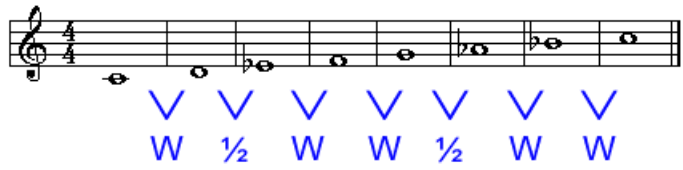


Figure 5: C Minor Scale



These two examples are the most basic scales, but there are many others which are commonly used in music. You might have heard of chromatic scales; these scales move by half steps only, thus using all 12 notes in the octave.

4 Circle of 5th's

The sequence of the scales move in fifths:

C would be our basic scale. no black notes

5 notes up the C scale would be G, thus G is the next scale. 1 black note: F#

5 notes up the G scale is D, thus D is the next scale. 2 black notes F# and C#

5 notes up the D scale is A, thus A is the next scale. 3 black notes, F# C# and G#

In other words the order is as follows:

This pattern continues until we have sharped all the notes, we then go into the flat keys, until get to the scale of F, which only has 1 flat. This is referred to as the Circle of Fifths, as described in the table below. This is basic knowledge to any piano player, but all we need to begin to see here is some of the amazing order that exists in the patterns of music. [6]

Scale	Sharps orFlats in Scale	Equivalent Scale	Flats in Scale
C Major	Only white keys.		
G Major	F#		
D Major	F# C#		
A Major	F# C# G#		
E Major	F# C# G# D#		
B Major	F# C# G# D# A#	Cb Major	Bb,Eb,Ab,Db,Gb,Cb,Fb
F# Major	F# C# G# D# A# E#	Gb Major	Bb,Eb,Ab,Db,Gb,Cb
C# Major	F# C# G# D# A# E# B#	Db Major	Bb,Eb,Ab,Db,Gb
Ab Major	Bb,Eb,Ab,Db		
Eb Major	Bb,Eb,Ab		
Bb Major	Bb,Eb		
F Major	Bb		

The terminology is as follows; if a music score uses the notes of the C Major scale then one says that it is in the key of C.

5 Chord Patterns

Chord structures follow various patterns as indicated below. The major chord is the first, third and fifth note of the scale. The Minor is similar, except it takes the third of the major chord and changes it to a half-step lower. The diminished chord moves the third and fifth

note a half-step down, while the augmented chord changes the fifth note to a half-step higher. There are many other chords structures for each key but the combinations and permutations remains the same from one key to the next.

Figure 5 describes some chord patterns.

5.1 Modulating

It is common for the key of a musical score to be different at various stages, but simply changing from one key to the next will not sound correct. One has to use one of the special methods for this to sound right. these methods are referred to as modulating. There are many methods of modulating one could use, but again, remember that the fascinating part about it is that the description will work in all the 12 keys, the difference being the starting point.

Below are three basic methods of modulating:

- a) Common chord modulation - a pivot chord is used to cross over, that is a chord which has notes that are common to both keys.
- b) Chromatic Modulation - here a certain portion of all 12 notes are used as a cross over.
- c) Direct Modulation - for example go from a G minor to a C major with the help of a "dominant six-five chord".

[6]

6 The Golden Ratio in Music

The golden ratio can be described to be that division of a line segment into two unequal segments such that the length "a", of the shorter segment, is to the length "b" of the longer, as the length of the longer is to the whole ($a + b$).

Figure 6 describes the Golden Ratio.

The golden ratio is found in the fibonacci sequence. It can also be observed in many of the patterns found in nature. It seems apparent that many composers of years gone by knew about this ratio and purposely used it in their compositions. For example Beethoven's Fifth Symphony seems to follow this pattern. The theory that the golden ratio is used in music is a disputed fact. John Putz of Alma College Michigan did a study on the works of Mozart and counted the measures of the first two movements of 28 of Mozart's three movement piano concertos. He counted the measures in the first movement, listed them and then counted the measures of the second movement and listed those alongside the first set of numbers. He then made a scatter plot of his findings and found that Mozart's concertos are within 94% certainty that they were purposely designed around the golden ratio. This could be one of the reasons why many people feel that Mozart's music is so delightful and memorable. John Putz put it very eloquently when he said that Mozart's

Figure 6: Chord Patterns

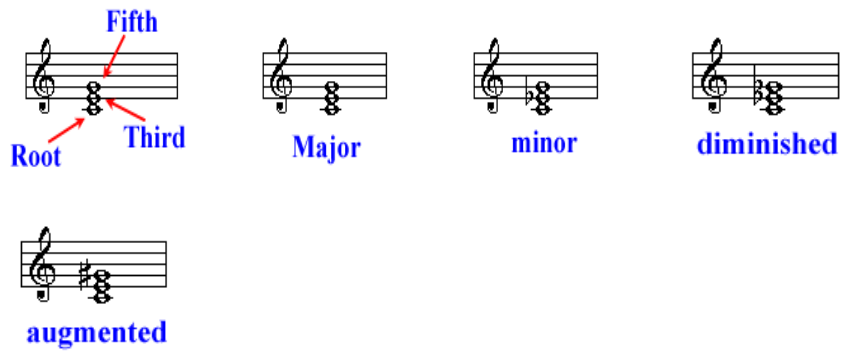


Figure 7: Golden Ratio

$$a/b = b/(a+b)$$



music manifests itself in form and balance. He also said that Mozart's music has been revered for its beauty and symmetrical proportions. In 1853 Henri Amiel said that "the balance of the whole is perfect". Hanns Dennerlein described Mozart's music as reflecting the "most exalted proportions" and that the composer himself had "an inborn sense for proportions". Eric Blom wrote that Mozart had "an infallible taste for saying exactly the right thing at the right time and at the right length. [7]

6.1 Music and Geometrical Patterns

The author and astrophysicist Mario Livio, in an interview with National Public Radio on his book titled *The Language of Symmetry* discussed how symmetry, transpositions and various patterns are found in music. Livio says that if one listens to Mozart's music the patterns can easily be heard. He pointed out that the symmetry is under translation and is similar to the patterns on a wallpaper design. He noted that sometimes Mozart changes a note here and there on purpose to get our attention. Livio also pointed out that if one puts a score on its side one can see the symmetry like putting a letter S on its side. [5]

Another interesting fact that Livio brought out was that the beauty of music and mathematics is that one doesn't need to read the entire encyclopedia on the subject to be creative. Therefore some of the best works of musicians have been done when they were young.

Livio believes that music we like best is the type that is in the middle of being completely predictable and a surprise. Sometimes, he says, that to get our attention an artist (in music and other art) will deviate from symmetry on purpose to get our attention, or to shock us. He paralleled this to the illustration of a person's reaction to someone whose glasses were complete tilted and skew. [5]

6.2 More on Mozart

Mozart is a very interesting character when it comes to math and music since he had a strong affinity towards math. According to his sister he studied math with much interest and wrote numbers all over the walls in his room, she also said that he gave himself entirely to the math and could speak of nothing else but figures. Another interesting fact is that in the margin of one of Mozart's scores is a calculation of the probability of winning the lottery. Mozart also experimented with random numbers in one of his compositions where the first 16 measures are played and then one would roll two dice and depending on the numbers that appeared on the dice one would play from a choice of 11 measures. The score was written in such a way that no matter which order one uses the music would blend together; thus creating a vast number of different effects. [5]

Mozart was born in Austria in 1756. He and his older sister were the only 2 of seven siblings who lived passed infancy. Mozart's father, who was a leading music teacher in Europe,

noticed his son's musical genius when he was only 3 years old. Mozart's father then gave the young Wolfgang Mozart intensive training in the organ, violin and harpsicord. Mozart wrote his first composition at the young age of 5 years old. Later his father and mother took him and his sister through Europe to show off their musical talent at various performance venues. This naturally helped Mozart in his musical career for the rest of his life. [7]

According to Mozart's sister, Mozart was very fascinated with Mathematics, gave himself to its study and spoke a lot about it. Although Mozart's salary was way above average for his time, he lived a very lavish life, spending far more than he could afford and was most often penniless and often ended up asking his friends for money. The exact reason for Mozart's death is not known and there is some speculation, but it seems as though he had influenza which turned into pneumonia and other related complications. Mozart died in Vienna in 1791 at the young age of 35 while writing a Requiem, which he believed was for his own impending death. [7]

7 Frequency

The most simplistic way to describe pitch is that it is the number of air vibrations per second. Thus middle C on the piano is $262Hz$ or 262 vibrations per second. The next C up from middle C is $262 * 2$ vibrations per second. For generality sake we will call the frequency of C, f . Then the next C would be $2^1 f$, and the next $2^2 f$, and so on. Notes that have differences in frequencies of $2^n f$, n an element of the Natural numbers, are perceived as the same as the original. We say these notes are separated by octaves. These notes are also equivalent to each other. [9]

On the piano, notes are in patterns of 12. Starting from the left the first white note is A, then B,C,D,E,F and G, then they start with A again and so on. The first note, A, has a very low sound with a frequency of $27.5Hz$, the next A up an octave has a frequency of $27.5(2)$, the next A has a frequency of $27.5(2^2)Hz$. This pattern continues until the 8th A, which has a frequency of $27.5(2^7)Hz$. [9]

The Pythagoreans observed that the vibrations and resulting sounds of a plucked string is raised an octave when the string is halved. This and other string length adjustments were studied by them. They also made observations with regard to intervals such as the interval of a 5th, like going from the note C to the note G. we count the notes of the C major scale. This interval is also the interval between the 2 twinkles of "Twinkle, Twinkle Little Star". They observed that that the 5th interval is produced when a string is lengthened from 1 unit to $3/2$ units. Sounds of notes were only related to the concept of frequency about 2000 years later by a French mathematician and priest Marin Mersenne.

1588-1648. Thus, if we call the frequency of C, f , then the frequency of G would be $3/2f$. [9]

The Pythagoreans studied the planets and the distances between them by using proportions so it was only natural for them to use proportions in their study of music. Thus the importance of music, to them, was how proportions pertained to the subject and that is why the use of string lengths became their focus. The discovery of intervals of a fourth, fifth, octave and double octave have been attributed to the Pythagoreans. The Pythagoreans found these ratios to be so significant that they developed a whole study of them. According to Gareth Loy they divided the scale into 12 equal parts as follows:

1. The octave is the ratio 12:6.
 2. The fifth is found by taking the arithmetic mean of the octave, defined as $x = (a + b)/2$. Thus $(12+6)/2=9$, and the ration 12:9=3:2 is the fifth.
 3. The fourth is found via the harmonic mean, defined as $x = 2ab/(a + b)$. Thus, $(2 * 12 * 6)/(12 + 6) = 8$ and the ratio 8 : 6 = 4 : 3 is the fourth.
- Pythagoras combined these results and come up with the harmonic propotion 12 : 9 = 8 : 6. Which he took to be the foundation of all music. [9]

Other intervals work as follows. The interval from G to the following C is a 4th and is obtained by $2f/((3/2)f) = 4/3f$. Therefore multiplying the frequency of a note by $4/3$ one can raise it by a 4th. Raising a note by a 3rd like from C to E, one would multiply the frequency by $5/4$. To raise a note by a 6th one would multiply the frequency by $5/3$. [9]

The following are the names of the keys of one octave on the piano and the numbers beneath represent the frequencies as described in this section.

C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C
f	$16/15f$	$9/8f$	$6/5f$	$5/4f$	$4/3f$	$64/45f$	$3/2f$	$8/5f$	$5/3f$	$16/9f$	$15/8f$	$2f$

[9]

8 Consonance and Dissonance

The following interesting observations are explained in Harkleroad's book, *The Math behind Music*. The idea is that when two notes are consonant they form a perfect interval and the interval that sounds almost harsh like playing 2 notes which are next to each other is referred to as dissonant.

If we take the frequency ratios of an octave 8 notes of the major scale and 12 notes when including all black and white notes and put them in a table as follows:

Intervals Ordered by Decreasing Consonance

Name		Ratio	Sum	Prime Factor	Limit
Perfect Intervals					
1	Unison	1/1	1 + 1 = 2	1	3-limit
2	Octave	2/1	2 + 1 = 3	2	
3	Fifth	3/2	3 + 2 = 5	3/2	
4	Fourth	4/3	4 + 3 = 7	2 ² /3	
Imperfect Intervals					
5	Major sixth	5/3	5 + 3 = 8	5/3	
6	Major third	5/4	5 + 4 = 9	5/2 ²	
7	Minor third	6/5	6 + 5 = 11	(2 * 3)/5	
8	Minor sixth	8/5	8 + 5 = 13	2 ³ /5	
Dissonant Intervals					
9	Major second	9/8	9 + 8 = 17	3 ² /2 ³	5-limit
10	Major seventh	15/8	15 + 8 = 23	(3 * 5)/2 ³	
11	Minor seventh	16/9	16 + 9 = 25	2 ⁴ /3 ²	
12	Minor second	16/15	16 + 15 = 31	2 ⁴ /(3 * 5)	
13	Tritone	64/45	64 + 45 = 109	2 ⁶ /(3 ² * 5)	

The first column represents the intervals starting at the most consonant sound, to the most dissonant. The third column represents the ratios of the frequencies. The fourth column gives the sum of the numerator and the denominator, notice how the sum increases as the dissonance increases. The fourth column has the frequency ratios factored. Notice how the prime factors range from 1 to 5. The perfect intervals with the most consonance have the lowest primes with the lowest exponents; the more dissonant the intervals the greater the factors and the higher the exponents. Consonance and dissonance are relative. In our present age we enjoy more dissonant sounds than people did a few hundred years ago, when imperfect intervals were regarded as dissonant and too modern. [8]

9 How to Vary a Theme Mathematically

Composers often take a series of notes, a basic tune, and vary it, exploring many angles of the tune or series of notes, which results in a beautiful composition. In his book *The Math Behind the Music*, Leon Harkleroad has some interesting observations on theme varying. In this section we will take a brief look at his study on Transposing, Retrograde and Inversions. [8]

9.1 Transposing

The most basic way to alter a tune is to raise it a pitch. The organist at sports events does this to heighten the emotions of the crowd, an example of which is seen in the figure below

Figure 9.1 is an excerpt from the music.

Another example is "Three Blind Mice", humm the tune and you will hear how it goes up a pitch.

Since there are 12 notes in an octave one can start at T_0 which indicates the starting pitch, raise the tune one semitone, which we indicate as T_1 . If the tune is raised T_{12} that would bring it to the original key, only one octave higher and would be classed as T_0 . Thus for n an element of the natural numbers and 12 divides n , then T_n would be classed as T_0 . The same applies to changing a tune from T_1 to T_{13} : Therefore in general for m and n elements of the whole numbers, with $m > n$ and $12/(m - n)$ there is no pitch difference in the change. In the course of a composition, when the tune is altered by T_1 , then later are T_3 results in a total of T_4 . Similarly, if a tune is raised to T_8 , then T_4 , the result is T_{12} which is the same as T_0 . Also T_5 and T_8 would result in T_{13} which is the same as T_1 . [8]

9.2 Retrograde

Another method composers use is called retrograde, where a tune is replayed in reverse order. For example "Auld Lang Syne":

Figure 9.2 describes Retrograde.

Observe the notes, F-A-C-D in "never bro't to mind" and compare it to the reverse order of D-C-A-F of "Should old acquaintance" [8]

9.3 Inversions

Another way to change a melody, besides flipping it left to right as in retrograde, is to flip it up-down. Here the melody is rewritten in such a way that every up step is rewritten as a down step and every down step is rewritten as an up step.

Inversions can be recognized more clearly by the listener than a retrograde.

An example of retrograde is "The Thunderer"

Figure 9.3 describes an inversion.

Here the left hand plays the inversion of the right hand. [8]

9.4 Combining Transpositions, Retrograde and Inversions

We saw that certain pitch changes result in the original pitch. Likewise a retrograde of a retrograde will result in the original melody. Also with inversions, an inversion of an inversion brings one back to the original tune. We denote retrograde with R and Inversion with I, we can then depict them as follows:

Figure 9: Retrograde

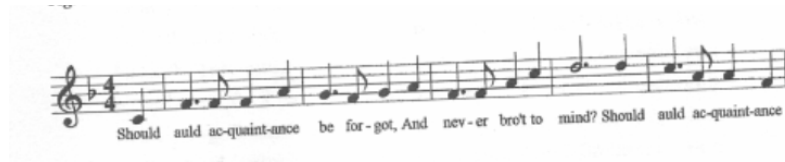


Figure 10: Inversion



Also note that $RR = T_0$ T_0 meaning the original tune
 $RT_n = T_nR$ for all n elements of whole numbers
 T_n meaning: Transposing n times

That is if you reverse the notes then change the pitch or first change the pitch then reverse the notes, you end up with the same result. Therefore these operations are commutative.

The figure below describes the following two changes: Retrograde - Transpose on the first row, then Transpose - Retrograde on the second row. Note how we end up with the same pattern.

Figure 9.4 describes the commutative property of retrograde and transposing.

Although we are using the word "inversion" an interesting observation is to note is that the word "opposite" is used in music and not the word "inverse". Let us look at the following sequence of changes that Leon Harkleroad describes in his book:

$$RT_5RT_2RT_8$$

Because of the commutative property we can do the following:

$$\begin{aligned} RT_5RT_2RT_8 &= T_5T_2T_8RRR \\ &= T_5T_2T_8T_0R && (RR = T_0) \\ &= T_3 && (T_5T_2T_8 = T_3) \end{aligned}$$

There are 24 Combinations of T's and R's: The 12 T_n 's and the 12 T_nR 's.
 Regarding Inversions we get 2 inversions resulting in the original melody.

$$\begin{aligned} \text{Therefore } II &= T_0 \\ IR &= RI \end{aligned}$$

The following is like a double reflection in a mirror.

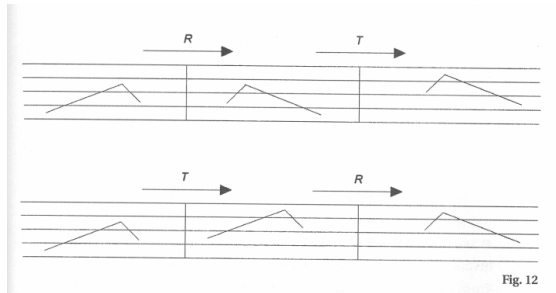
$$IT_n = T_{12-n}I$$

In general, the following operations can be identified (where T_0 is the identity element for any operation X);

$$\begin{aligned} XT_0 = T_0X = X \quad \text{||| For every operation X there is a Y such that} \\ SY = YX = T_0 \end{aligned}$$

For example: $T_5T_7 = T_0 = T_7T_5$ For any operations X, Y and Z we have:
 $(T_2I)(T_2I) = T_0$

Figure 11: Commutative



$$(XY)Z = X(YZ)$$

For example: if $X = T_9R$ and $T = T_5IR$ and $Z = T_3$

Then:

$$\begin{aligned} [(T_9R)(T_5IR)]T_3 &= [T_2I]T_3 \\ &= T_{11}I \\ &= (T_9R)[(T_5IR)T_3] \\ &= T_9R[T_2IR] \\ &= T_{11}I \end{aligned}$$

[8]

10 A Few General Patterns for Improvising Music

Improvising is the art of spontaneously creating music while playing. Improvising is incorporating all kinds of known patterns while playing an ordinary piece of music. In short it is the sum of everything one has heard. Due to the constraints of timing, length and placement, to name a few, improvising is a technique that requires much skill and mathematical understanding. [10]

To improvise a full measure determine the distance from Point A to Point B the first beat to start and the last note to land on, then select the appropriate number of 1/16 notes with the appropriate fill. [10]

In the following diagram observe the step up or step down in the first measure and the matching two-beat fills. This is just an example of how music can be improvised, only a sample of what can be done is shown below, since, by simple virtue of the magnitude of the amount of combinations and permutations an explanation of everything would fill a book.

10.1 Some Techniques for Improvising

The following diagram is from a comprehensive book on Improvising by Gail Smith of Mel Bay Music Publishing. Assume the added calculating starts on the first note and ends on the desired note:











Figure 10.1 illustrates some improvisation.

[10]

10.2 Conclusion

Most Musicians are aware of the tie between Math and Music, but unless one really studies this amazing tie one will never really see the depths of the connection. We have just touched the tip of the iceberg in the subject of math versus music. Each section that we looked at can lead us deeply into the mysteries of the mathematical connections of music.

Figure 12: Improvising

 <p>Step Up</p>	+4+4+1
 <p>Step Down</p>	-4-4+1
 <p>Up a third</p>	-4+4+1
 <p>Down a third</p>	+4-4-1
 <p>Up a fourth</p>	jump-4 (a 5 th down) +3+4+1
 <p>Down a fourth</p>	jump+4 (a 5 th up) -3-4+1
 <p>Up a fifth</p>	-3+1+4+1
 <p>Down a fifth</p>	+3-1-4-1
 <p>Up a sixth</p>	(a 3 rd down) +3+4+1
 <p>Down a sixth</p>	(Smith)

The more a Musician can study in the area of how mathematics is tied in with music then the more understanding he/she can enjoy in the beauty of both music and mathematics.

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