

# Cyclic projectors and separation theorems in idempotent convex geometry\*

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## Abstract

Semimodules over idempotent semirings like the max-plus or tropical semiring have much in common with convex cones. This analogy is particularly apparent in the case of subsemimodules of the  $n$ -fold cartesian product of the max-plus semiring: it is known that one can separate a vector from a closed subsemimodule that does not contain it. We establish here a more general separation theorem, which applies to any finite collection of closed subsemimodules with a trivial intersection. In order to prove this theorem, we investigate the spectral properties of certain nonlinear operators called here idempotent cyclic projectors. These are idempotent analogues of the cyclic nearest-point projections known in convex analysis. The spectrum of idempotent cyclic projectors is characterized in terms of a suitable extension of Hilbert's projective metric. We deduce as a corollary of our main results the idempotent analogue of Helly's theorem.

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## 1 Introduction

Some nonlinear problems in optimization theory and mathematical physics turn out to be linear over semirings with an idempotent addition  $\oplus$  [1], [8], [14]. We recall that the idempotency of  $\oplus$  means  $a \oplus a = a$  for all  $a$ , and that the role of this addition is most often played

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by the operations of taking maxima or minima. The search for idempotent analogues of classical results has motivated the development of idempotent mathematics, see the recent collection of articles [16] and also [15] for more background.

One of the most studied idempotent semirings is the max-plus semiring. It is the set  $\mathbf{R} \cup \{-\infty\}$  equipped with the operations of addition  $a \oplus b := \max(a, b)$  and multiplication  $a \odot b := a + b$ . The zero element  $\mathbf{0}$  of this semiring is equal to  $-\infty$ , and the semiring unity  $\mathbf{1}$  is equal to 0. Some algebraic structures which coincide with the max-plus semiring (up to isomorphism) have appeared under other names. In particular, the min-plus or tropical semiring is obtained by replacing  $-\infty$  by  $+\infty$  and  $\max(a, b)$  by  $\min(a, b)$  above. Applying  $x \mapsto \exp(x)$  to the max-plus semiring (assume  $\exp(-\infty) = 0$ ), we obtain the max-times semiring, further denoted by  $\mathbf{R}_{\max, m}$ . It is the set of nonnegative numbers ( $\mathbf{R}_+$ ), equipped with the operations  $a \oplus b = \max(a, b)$  and  $a \odot b = a \times b$ . The zero and unit elements of  $\mathbf{R}_{\max, m}$  coincide with the usual 0 and 1. Our main results (Sect. 4) will be stated over this semiring, as it makes clearer some analogies with classical convex analysis.

We shall consider here subsemimodules of the  $n$ -fold cartesian product  $\mathcal{K}^n$  of a semiring  $\mathcal{K}$  and, more generally, of the set  $\mathcal{K}^I$  of functions from a set  $I$  to  $\mathcal{K}$ . Further examples can be found e.g. in [1], [14] and [17].

In an idempotent semiring, there is a canonical order relation, for which every element is “nonnegative”. Therefore, idempotent semimodules have much in common with the semimodules over the semiring of nonnegative numbers, that is, with *convex cones* [19]. One of the first results based on this idea is the separation theorem for convex sets over “extremal algebras” proved by K. Zimmermann in [21]. This theorem implies that a point in  $\mathbf{R}_{\max, m}^n$ , which does not belong to a semimodule that is closed in the Euclidean topology, can be separated from it by an idempotent analogue of a closed halfspace. Generalizations of this result were obtained in a work by S.N. Samborskiĭ and G.B. Shpiz [20] and in works by G. Cohen, J.-P. Quadrat, I. Singer, and the first author [5], [6]. In the special case of finitely generated semimodules, a separation theorem has also been obtained by M. Develin and B. Sturmfels in [10], with a strong emphasis on some combinatorial aspects of the result.

The main result of this paper, Theorem 21, shows that *several* closed semimodules which do not have common nonzero points can be separated from each other. This means that for each of these semimodules, we can select an idempotent halfspace containing it, in such a way that these halfspaces also do not have common nonzero points.

Even in the case of two semimodules, this statement has not been proved in the idempotent literature. Indeed, the earlier separation theorems deal with the separation of a point from an (idempotent) convex set or semimodule, rather than with the separation of two convex

sets or semimodules. Note that unlike in the classical case, separating two convex sets cannot be reduced to separating a point from a convex set. More precisely, it is easily shown that two convex sets  $A$  and  $B$  can be separated if and only if the point  $0$  can be separated from their Minkowski difference  $A - B$ , in classical convex geometry. In idempotent geometry, an analogue of Minkowski difference can still be defined, consider  $A \ominus B = \{x \mid \exists b \in B : x \oplus b \in A\}$ . However, due to the idempotency of the addition, we cannot recover a halfspace separating  $A$  and  $B$  from a halfspace separating  $0$  from  $A \ominus B$ .

In order to prove the main result, Theorem 21, we investigate the spectral properties of idempotent cyclic projectors. By idempotent cyclic projectors we mean finite compositions of certain nonlinear projectors on idempotent semimodules. The continuity and homogeneity of these nonlinear projectors enables us to apply to their compositions, i.e. to the cyclic projectors, some results from non-linear Perron-Frobenius theory. The main idea is to prove the equivalence of the following three statements: 1) that the semimodules have trivial intersection, 2) that the separating halfspaces exist, and 3) that the spectral radius of the associated cyclic projector is strictly less than 1. This equivalence is established in Theorems 16 and 19, which deal with the special case of archimedean semimodules, i.e. semimodules containing at least one positive vector. As an ingredient of the proof, we use a nonlinear extension of Collatz-Wielandt's theorem obtained by R.D. Nussbaum [18]. To derive the main separation result, Theorem 21, we show that for any collection of trivially intersecting semimodules, there is a collection of trivially intersecting *archimedean* semimodules, such that every semimodule from the first collection is contained in an archimedean semimodule from the second collection.

We also show in Theorems 13 and 15 that the orbit of an eigenvector of a cyclic projector maximizes a certain objective function. We call this maximum the Hilbert value of semimodules, as it is a natural generalization of Hilbert's projective metric, and characterize the spectrum of cyclic projectors in terms of these Hilbert values (Theorem 18).

The projectors on idempotent semimodules, which constitute the cyclic projectors considered here, have been studied by R.A. Cuninghame-Green, see [7] and [8], Chapter 8, where they appear as  $AA^*$ -products. The geometrical properties of these projectors have been used in [5, 6] to establish separation theorems. The same operators have also been studied by G.L. Litvinov, V.P. Maslov and G.B. Shpiz, who obtained in [17] idempotent analogues of several results from functional analysis, including the analytic form of the Hahn-Banach theorem.

The idempotent cyclic projectors have been introduced, in the case of two semimodules, by P. Butkovič and R.A. Cuninghame-Green in [9], where these operators give rise to an efficient

(pseudo-polynomial) algorithm for finding a point in the intersection of two finitely generated subsemimodules of  $\mathbf{R}_{\max, m}^n$ . In convex analysis and optimization theory, an analogous role is played by the cyclic nearest-point projections on convex sets [3].

As a corollary of Theorems 19 and 21, we deduce a max-plus analogue of Helly's theorem. This result has also been obtained, with a different proof, by F. Meunier and the first author [13].

Our main results apply to subsemimodules of  $\mathbf{R}_{\max, m}^n$ . Some of our results still hold in a more general setting, see Sect. 3. However, the separation of several semimodules in such a generality remains an open question.

The results of this paper are presented as follows. Sect. 2 describes the main assumptions that are satisfied by the semimodules of the paper. Besides that, it is occupied by some basic notions and facts that will be used further. Sect. 3 is devoted to the results obtained in the most general setting, with respect to the assumptions of Sect. 2. The main results for the case  $\mathbf{R}_{\max, m}^n$  are obtained in Sect. 4. These results include separation of several semimodules and characterization of the spectrum of cyclic projectors.

## 2 Preliminary results on projectors and separation

We start this section with some details concerning the role of partial order in idempotent algebraic structures. For more background, we refer the reader to e.g. [1, 8, 17].

The idempotent addition  $\oplus$  defines the canonical order relation  $\leq_{\oplus}$  on the semiring  $\mathcal{K}$  by the rule  $\lambda \oplus \mu = \mu \Leftrightarrow \lambda \leq_{\oplus} \mu$  for  $\lambda, \mu \in \mathcal{K}$ . The idempotent sum  $\lambda \oplus \mu$  is equal to the least upper bound  $\sup(\lambda, \mu)$  with respect to the order  $\leq_{\oplus}$ . The idempotent sum of an arbitrary subset is defined to be the least upper bound of this subset, if this least upper bound exists. In a semimodule  $\mathcal{V}$ , we define the order relation  $\leq_{\oplus, \mathcal{V}}$  in the same way. The relation  $\lambda \leq_{\oplus} \mu$  between  $\lambda, \mu \in \mathcal{K}$  implies  $\lambda x \leq_{\oplus, \mathcal{V}} \mu x$  for all  $x \in \mathcal{V}$ . When  $\mathcal{V} = \mathcal{K}^n$  and  $\mathcal{K} = \mathbf{R}_{\max, m}$ , the order  $\leq_{\oplus}$  coincides with the usual linear order on  $\mathbf{R}_+$ , and the order  $\leq_{\oplus, \mathcal{V}}$  coincides with the standard pointwise order on  $\mathbf{R}^n$ . For this reason, we will write  $\leq$  instead of  $\leq_{\oplus}$  and  $\leq_{\oplus, \mathcal{V}}$ , in the sequel.

A semiring or a semimodule will be called *b-complete* (see [17]), if it is closed under the sum (i.e. the supremum) of any subset bounded from above, and the multiplication distributes over such sums. If the least upper bound  $\oplus$  exists for all subsets bounded from above, then the greatest lower bound  $\wedge$  exists for all subsets bounded from below. Consequently, the

greatest lower bound exists for any subset of a  $b$ -complete semiring or a semimodule, since such a subset is bounded from below by  $\mathbf{0}$ .

Also note that if  $\mathcal{K}$  is a  $b$ -complete semiring, and the set  $\mathcal{K} \setminus \{\mathbf{0}\}$  is a multiplicative group, then this group is abelian by Iwasawa's theorem [2]. A semiring  $\mathcal{K}$  such that the set  $\mathcal{K} \setminus \{\mathbf{0}\}$  is an abelian multiplicative group is called an *idempotent semifield*.

We shall consider semirings  $\mathcal{K}$  and semimodules  $\mathcal{V}$  over  $\mathcal{K}$  that satisfy the following assumptions:

(A0): the semiring  $\mathcal{K}$  is a  $b$ -complete idempotent semifield, and the semimodule  $\mathcal{V}$  is a  $b$ -complete semimodule over  $\mathcal{K}$ ;

(A1): for all elements  $x$  and  $y \neq \mathbf{0}$  from  $\mathcal{V}$ , the set  $\{\lambda \in \mathcal{K} \mid \lambda y \leq x\}$  is bounded from above.

Assumptions (A0, A1) imply that the operation

$$x/y = \max\{\lambda \in \mathcal{K} \mid \lambda y \leq x\}. \quad (1)$$

is defined for all elements  $x$  and  $y \neq \mathbf{0}$  from  $\mathcal{V}$ . The following can be viewed as another definition of the operation  $/$  equivalent to (1):

$$\lambda y \leq x \Leftrightarrow \lambda \leq x/y. \quad (2)$$

In the case  $\mathcal{V} = \mathcal{K}^I$ ,

$$x/y = \bigwedge_{i: y_i \neq \mathbf{0}} x_i/y_i. \quad (3)$$

The operation  $/$  has the following properties:

$$\left(\bigwedge_{\alpha} x_{\alpha}\right)/y = \bigwedge_{\alpha} (x_{\alpha}/y), \quad (x/\bigoplus_{\alpha} y_{\alpha}) = \bigwedge_{\alpha} (x/y_{\alpha}) \quad (4)$$

$$(\lambda x)/y = \lambda(x/y) \quad \forall \lambda, \quad y/(\lambda x) = \lambda^{-1}(y/x) \quad \forall \lambda \neq \mathbf{0}. \quad (5)$$

We also need the following lemma.

**Lemma 1** *Under (A0, A1),  $x/x = \mathbf{1}$  for all nonzero vectors  $x \in \mathcal{V}$ . If  $\lambda x = x$  for a nonzero vector  $x \in \mathcal{V}$ , then  $\lambda = \mathbf{1}$ .*

*Proof.* The inequality  $x \leq x$  implies that  $x/x \geq \mathbf{1}$ , see (1). On the other hand, we have that  $(x/x)x \leq x$ . Multiplying this by  $x/x$ , we obtain that  $(x/x)^2x \leq (x/x)x \leq x$ , hence  $(x/x)^2 \leq x/x$  and  $x/x \leq \mathbf{1}$ . Thus  $x/x = \mathbf{1}$ .

If  $\lambda x = x$  for some  $x \neq \mathbf{0}$ , then  $\lambda(x/x) = (\lambda x)/x = x/x$  and so  $\lambda = \mathbf{1}$ .  $\square$

**Definition 2** A subsemimodule  $V$  of  $\mathcal{V}$  is a  $b$ -(sub)semimodule, if  $V$  is closed under the sum of any of its subsets bounded from above in  $\mathcal{V}$ .

Let  $V$  be a  $b$ -subsemimodule of the semimodule  $\mathcal{V}$ . Consider the operator  $P_V$  defined by

$$P_V(x) = \max\{u \in V \mid u \leq x\}, \quad (6)$$

for every element  $x \in \mathcal{V}$ . Here we use “max” to indicate that the least upper bound belongs to the set. The operator  $P_V$  is a *projector* onto the subsemimodule  $V$ , as  $P_V(x) \in V$  for any  $x \in \mathcal{V}$  and  $P_V(v) \in V$  for any  $v \in V$ . In principle,  $P_V$  can be defined for all subsets of  $\mathcal{V}$ , if we write sup instead of max in (6), but then  $P_V$  may not be a projector on  $V$ .

**Definition 3** A subsemimodule  $V$  of  $\mathcal{V}$  is called elementary, if  $V = \{\lambda y \mid \lambda \in \mathcal{K}\}$  for some  $y \in \mathcal{V}$ . The projector onto such a semimodule is also called elementary.

Assumptions (A0, A1) imply that elementary semimodules are  $b$ -semimodules. For the elementary semimodule  $V = \{\lambda y \mid \lambda \in \mathcal{K}\}$ , the projector  $P_V$  is given by  $P_V(x) = (x/y)y$ , and this fact can be generalized as follows.

**Proposition 4** If  $V$  is a  $b$ -subsemimodule of  $\mathcal{V}$  and  $P_V(x) = \lambda y$  for some  $\lambda \in \mathcal{K}$  and  $x, y \in \mathcal{V}$ , then  $P_V(x) = (x/y)y$ .

*Proof.* If  $V$  is a  $b$ -semimodule, then  $y \in V$ , and  $(x/y)y \leq x$  implies that  $(x/y)y \leq P_V(x) = \lambda y$ . On the other hand,  $\lambda y \leq x$  implies  $\lambda \leq x/y$  and  $\lambda y \leq (x/y)y$ .  $\square$

Note that  $P_V$  is isotone with respect to inclusion:

$$U \subset V \Rightarrow P_U(x) \leq P_V(x) \text{ for all } x. \quad (7)$$

It is also homogeneous and isotone:

$$P_V(\lambda x) = \lambda P_V(x), \quad x \leq y \Rightarrow P_V(x) \leq P_V(y). \quad (8)$$

We remark that the operator  $P_V$  is in general *not* linear with respect to  $\oplus$  or  $\wedge$  operations, even in the case  $\mathcal{V} = \mathbf{R}_{\max, m}^n$ .

In idempotent geometry, the role of halfspace is played by the following object.

**Definition 5** A set  $H$  given by

$$H = \{x \mid u/x \geq v/x\} \cup \{\mathbf{0}\} \quad (9)$$

with  $u, v \in \mathbf{R}_{\max, m}^n$ ,  $u \leq v$ , will be called (idempotent) halfspace.

Properties (4) and (5) of the operation  $/$  imply that any halfspace is a semimodule. If  $\mathcal{V} = \mathcal{K}^I$ , then we can use (3) and then

$$H = \{x \mid \bigwedge_{i: x_i \neq \mathbf{0}} u_i x_i^{-1} \geq \bigwedge_{i: x_i \neq \mathbf{0}} v_i x_i^{-1}\} \cup \{\mathbf{0}\}. \quad (10)$$

If  $\mathcal{V} = \mathcal{K}^n$  and all coordinates of  $u$  and  $v$  are nonzero, then we have that

$$H = \{x \mid \bigoplus_{1, \dots, n} x_i u_i^{-1} \leq \bigoplus_{1, \dots, n} x_i v_i^{-1}\}. \quad (11)$$

Such idempotent halfspaces formally resemble the closed homogeneous halfspaces of the finite-dimensional convex geometry [19].

Since the operation  $/$  is isotone with respect to the first argument, we can replace the inequalities in (5), (10) and (11) by the equalities. For instance, definition (5) can be rewritten as

$$H = \{x \mid u/x = v/x\} \cup \{\mathbf{0}\}, \quad (12)$$

where  $u \leq v$ .

The present paper is concerned with the separation of several  $b$ -semimodules, whereas the separation theorems which have been established previously, like the ones of [5, 6], deal with the separation of one point from a semimodule. For the convenience of the reader, we next state a theorem, which is a variant of a separation theorem of [5]. The difference is in that we deal with  $b$ -complete semimodules rather than with complete semimodules. Both results are closely related with the idempotent Hahn-Banach theorem of [17].

**Theorem 6** (Compare with [5], Theorem 8) *Let  $V$  be a  $b$ -subsemimodule of  $\mathcal{V}$  and let  $u \notin V$ . Then the halfspace*

$$H = \{x \mid P_V(u)/x \geq u/x\} \cup \{\mathbf{0}\} \quad (13)$$

*contains  $V$  but not  $u$ .*

*Proof.* Take a nonzero vector  $x \in V$  (the case  $x = \mathbf{0}$  is trivial). Since  $(u/x)x \leq u$ , we have  $(u/x)x \leq P_V(u)$ , which is by (2) equivalent to  $u/x \leq P_V(u)/x$ . Hence  $V \subseteq H$ .

Take  $x = u$  and assume that  $P_V(u)/u \geq u/u = \mathbf{1}$ . This is equivalent to  $u \leq P_V(u)$  and hence to  $u = P_V(u)$ . Since  $V$  is a  $b$ -semimodule, we have that  $u \in V$ , which is a contradiction. Hence  $u \notin H$ .  $\square$

**Definition 7** *Consider the preorder relation  $\preceq$  defined by*

$$x \preceq y \Leftrightarrow y/x > \mathbf{0}. \quad (14)$$

We say that  $x$  and  $y$  are comparable, and we write  $x \sim y$ , if  $x \preceq y$  and  $y \preceq x$ . Equivalently,

$$x \sim y \Leftrightarrow (x/y)(y/x) > \mathbf{0}. \quad (15)$$

Note that if  $y = \lambda x$  with  $\lambda \neq \mathbf{0}$ , then  $y \sim x$ , and that the inequality  $x \leq y$ , if  $x \neq \mathbf{0}$ , implies that  $x \preceq y$ . In particular,  $P_V(x) \preceq x$  for any nonzero  $x \in \mathcal{V}$  and any semimodule  $V$ , provided that  $P_V(x)$  is nonzero.

When  $\mathcal{V} = \mathcal{K}^n$ , comparability can be characterized in terms of supports. Recall that the *support* of a vector  $x$  in  $\mathcal{K}^n$  is defined by  $\text{supp}(x) = \{i \mid x_i \neq 0\}$ . It can be checked that for all  $x, y \in \mathcal{K}^n$ , we have  $x \preceq y$  iff  $\text{supp}(x) \subset \text{supp}(y)$ , and so,  $x \sim y$  iff  $\text{supp}(x) = \text{supp}(y)$ .

**Proposition 8** *Let  $x \in \mathcal{V}$  be a nonzero vector and let  $V \subseteq \mathcal{V}$  be a  $b$ -semimodule containing a nonzero vector  $y$ . If  $y \preceq x$ , then  $P_V(x)$  is nonzero, and  $y \preceq P_V(x) \preceq x$ . If  $y \sim x$ , then  $P_V(x) \sim x$ .*

*Proof.* By the definition of  $/$  and by (14), there exists  $\alpha$  such that  $\alpha y \leq x$ . Then  $\alpha y \leq P_V(x)$ , hence  $P_V(x)$  is nonzero and  $y \preceq P_V(x)$ .  $\square$

**Proposition 9** *Let  $F$  be an isotone and homogeneous operator, let  $\lambda, \mu$  be arbitrary scalars from  $\mathcal{K}$  and let  $v$  and  $u$  be nonzero vectors such that  $v \prec u$ . Suppose that one of the following is true:*

1.  $Fv \geq \mu v$  and  $Fu = \lambda u$ ;
2.  $Fv = \mu v$  and  $Fu \leq \lambda u$ .

Then  $\mu \leq \lambda$ .

*Proof.* Applying  $F$  to the inequality  $(u/v)v \leq u$  and using any of the given conditions, we obtain that  $(u/v)\mu v \leq \lambda u$ . If  $\lambda = \mathbf{0}$ , then  $\mu = \mathbf{0}$ . If  $\lambda$  is invertible, then by (2)  $(u/v)\mu \lambda^{-1} \leq u/v$ . Cancelling  $u/v$ , we get  $\mu \leq \lambda$ .  $\square$

Properties (4) and (5) imply that the sets  $\{x \mid x \preceq y\}$ ,  $\{x \mid x \succeq y\}$  and hence  $\{x \mid x \sim y\}$  are subsemimodules of  $\mathcal{V}$ . For any semimodule  $V \subset \mathcal{V}$  and any vector  $y \in \mathcal{V}$ , we define

$$V^y = \{x \in V \mid x \preceq y\}, \quad (16)$$

which is a subsemimodule of  $V$ . When  $\mathcal{V} = \mathcal{K}^n$ ,  $V^y$  is uniquely determined by the support  $M$  of  $y$ . For this reason, for all  $M \subseteq \{1, \dots, n\}$ , we set

$$V^M = \{x \in V \mid \text{supp}(x) \subset M\}. \quad (17)$$

**Definition 10** A vector  $x \in \mathcal{V}$  is called archimedean, if  $y \preceq x$  for all  $y \in \mathcal{V}$ . A semimodule  $V \subseteq \mathcal{V}$  is called archimedean, if it contains an archimedean vector. A halfspace will be called archimedean, if both vectors defining it (e.g.  $u$  and  $v$  in (9)) are archimedean.

Of course, Def. 10 makes sense only in the case when  $\mathcal{V}$  satisfies the following assumption:

(A2): the semimodule  $\mathcal{V}$  has an archimedean vector.

This assumption is satisfied by the semimodules  $\mathcal{V} = \mathcal{K}^n$  (we are also assuming (A0, A1)).

In this case, archimedean halfspaces have been written explicitly in (11).

### 3 Cyclic projectors and separation theorems: general results

In this section we study cyclic projectors, that is, compositions of projectors

$$P_{V_k} \cdots P_{V_1}, \quad (18)$$

where  $V_1, \dots, V_k$  are  $b$ -subsemimodules of  $\mathcal{V}$ . We assume (A0, A1), which means in particular that  $\mathcal{K}$  is an idempotent semifield, and state general results concerning cyclic projectors and separation properties. For the notational convenience, we will write  $P_l$  instead of  $P_{V_l}$ . We will also adopt a convention of cyclic numbering of indices of projectors and semimodules, so that  $P_{l+k} = P_l$  and  $V_{l+k} = V_l$  for all  $l$ .

**Definition 11** Let  $x^1, \dots, x^k$  be nonzero elements of  $\mathcal{V}$ . The value

$$d_H(x^1, \dots, x^k) = (x^1/x^2) (x^2/x^3) \dots (x^k/x^1). \quad (19)$$

will be called the Hilbert value of  $x^1, \dots, x^k$ .

It follows from Def. 7 that  $d_H(x^1, \dots, x^k) \neq \mathbf{0}$  if and only if all vectors  $x^1, \dots, x^k$  are comparable. One can show that  $d_H(x^1, \dots, x^k) \leq \mathbf{1}$ . This inequality is an equality if and only if  $x^1, \dots, x^k$  differ from each other only by scalar multiples. The Hilbert value is invariant under multiplication of any of its arguments by an invertible scalar, and by cyclic permutation of its arguments.

The Hilbert value of two vectors  $x^1, x^2$  was studied in [5]. For two comparable vectors in  $\mathbf{R}_{\max, m}^n$ , that is, for two vectors with common support  $M$  it is given by

$$d_H(x^1, x^2) = \min_{i, j \in M} (x_i^1 (x_i^2)^{-1} x_j^2 (x_j^1)^{-1}), \quad (20)$$

so that  $-\log(d_H(x^1, x^2))$  coincides with Hilbert's projective metric

$$\delta_H(x^1, x^2) = \log(\max_{i,j \in M} (x_i^1 (x_i^2)^{-1} x_j^2 (x_j^1)^{-1})) = -\log(d_H(x^1, x^2)). \quad (21)$$

**Definition 12** *The Hilbert value of  $k$  subsemimodules  $V_1, \dots, V_k$  of  $\mathcal{V}$  is defined by*

$$d_H(V_1, \dots, V_k) = \sup_{x^1 \in V_1, \dots, x^k \in V_k} d_H(x^1, \dots, x^k) \quad (22)$$

**Theorem 13** *Suppose that the operator  $P_k \cdots P_1$  has an eigenvector  $y$  with eigenvalue  $\lambda$ . Then*

$$\lambda = \max_{x^1 \in V_1^y, \dots, x^k \in V_k^y} d_H(x^1, \dots, x^k) = d_H(\bar{x}^1, \dots, \bar{x}^k), \quad (23)$$

where  $\bar{x}^i = P_i \cdots P_1 y$ .

*Proof.* Note that  $\bar{x}^i$ , for any  $i$ , is an eigenvector of  $P_{i+k} \cdots P_{i+1}$  and that all these vectors are comparable with  $y$ . Further, let  $x^1, \dots, x^k$  be arbitrary elements of  $V_1^y, \dots, V_k^y$ , respectively, and let  $\alpha_1, \dots, \alpha_k$  be scalars such that

$$\begin{aligned} \alpha_1 x^2 &\leq P_2 x^1, \\ &\vdots \\ \alpha_{k-1} x^k &\leq P_k x^{k-1}, \\ \alpha_k x^1 &\leq P_1 x^k, \end{aligned} \quad (24)$$

Take the last inequality. Applying  $P_2$  to both sides and using the first inequality, we have that  $\alpha_1 \alpha_k x^2 \leq P_2 P_1 x^k$ . Further, we apply  $P_3$  to this inequality and use the inequality  $\alpha_2 x^3 \leq P_3 x^2$ . Proceeding in the same manner, we finally obtain

$$\alpha_1 \dots \alpha_k x^k \leq P_k \cdots P_1 x^k. \quad (25)$$

It follows from Prop. 9 that  $\alpha_1 \cdots \alpha_k \leq \lambda$ . We take  $\alpha_i = x^i / x^{i+1}$  for  $i = 1, \dots, k-1$ , and  $\alpha_k = x^k / x^1$ . This leads to

$$d_H(V_1^y, \dots, V_k^y) \leq \lambda. \quad (26)$$

Note that this inequality is true if  $V_1, \dots, V_k$  are not  $b$ -semimodules. Applying Prop. 4 we have that  $\lambda y = d_H(\bar{x}^1, \dots, \bar{x}^k) y$ . By Lemma 1 we can cancel  $y$ , and the observation that  $\bar{x}^i \in V_i^y$  for all  $i$  yields the desired equality.  $\square$

The situation when  $P_k \cdots P_1$  has an eigenvector with nonzero eigenvalue occurs, if at least one of the semimodules  $V_1, \dots, V_k$  is elementary, that is, generated by a single vector  $x^i$ , and if all other semimodules have vectors comparable with  $x^i$ . In this case  $P_k \cdots P_{i+1} x^i$  is the only eigenvector of  $P_k \cdots P_1$  with nonzero eigenvalue.

To obtain the following lemma, we use Prop. 8.

**Lemma 14** Let  $x^1 \in V^1$  and  $x^i = P_i x^{i-1}$  for  $i = 2, \dots, k$ . Then, the Hilbert value  $d_H(x^1, \dots, x^k)$  is not equal to  $\mathbf{0}$  if and only if  $V_2, \dots, V_k$  have vectors comparable with  $x^1$ .

**Theorem 15** Suppose that the vectors  $x^i$ ,  $i = 1, \dots$  are such that  $x^1 \in V_1$  and  $x^i = P_i x^{i-1}$  for  $i = 2, \dots$ . Then  $d_H(x^{l+1}, \dots, x^{l+k})$  is nondecreasing with  $l$  so that the following inequalities hold for all  $l$ :

$$d_H(x^1, \dots, x^k) \leq d_H(x^{l+1}, \dots, x^{l+k}) \leq \mathbf{1}. \quad (27)$$

*Proof.* As  $V_i$  are  $b$ -semimodules,  $x^i \in V_i$  for all  $i$ . If the Hilbert value is  $\mathbf{0}$  for all  $l$ , then there is nothing to prove. So we assume that there exists a least  $l = l_{\min}$  for which the Hilbert value  $d_H(x^l, \dots, x^{l+k-1})$  is nonzero. As it is nonzero, by Lemma 14, all  $x^l, \dots, x^{l+k-1}$  are comparable. By Prop. 8,  $x^{l+k}$  is also comparable with them, and the same is true about the rest of the sequence, hence  $d_H(x^l, \dots, x^{l+k-1})$  is nonzero for all  $l \geq l_{\min}$ . Now we take any  $l \geq l_{\min}$  and consider the composition

$$P_{l+k} P'_{l+k-1} \cdots P'_{l+1}, \quad (28)$$

where  $P'_i$ , for  $i = l+1, \dots, l+k-1$ , are elementary projectors onto the semimodules generated by  $x^i$ . The operator (28) has an eigenvector  $x^{l+k}$ . By Theorem 13

$$d_H(x^l, \dots, x^{l+k-1}) \leq \max_{y \in V_i, y \preceq x^{l+k}} d_H(x^{l+1}, \dots, x^{l+k-1}, y) = d_H(x^{l+1}, \dots, x^{l+k}). \quad (29)$$

for all  $l = 1, \dots$   $\square$

The following theorem assumes the existence of archimedean vectors (A2).

**Theorem 16** Suppose that  $P_k \cdots P_1$  has an archimedean eigenvector  $y$  with nonzero eigenvalue  $\lambda$ . The following are equivalent:

1. there exist an archimedean vector  $x$  and a scalar  $\mu < \mathbf{1}$  such that  $P_k \cdots P_1 x \leq \mu x$ ;
2. for all  $i = 1, \dots, k$  there exist archimedean halfspaces  $H_i$  such that  $V_i \subseteq H_i$  and  $H_1 \cap \cdots \cap H_k = \{\mathbf{0}\}$ ;
3.  $V_1 \cap \cdots \cap V_k = \{\mathbf{0}\}$ ;
4.  $\lambda < \mathbf{1}$ .

*Proof.* 1  $\Rightarrow$  2: Denote  $x^0 = x$  and  $x^i = P_i \cdots P_1 x^0$ . Note that all the  $x^i$  are also archimedean by Prop. 8. For all  $i = 1, \dots, k$  we have that

$$V_i \subseteq \{u: x^{i-1}/u = x^i/u\} = H_i. \quad (30)$$

Indeed, if  $x^{i-1} = x^i$ , then  $H_i$  coincides with the whole  $\mathcal{V}$ . If  $x^{i-1} \neq x^i$ , which means that  $x^i \notin V_{i-1}$ , then the inclusion in (30) follows from Theorem 6. Assume that there exists a nonzero vector  $u$  which belongs to every  $H_i$ . Then  $x^k/u = x/u$ . But  $x^k/u \leq (\mu x)/u \leq x/u$ , hence  $\mu(x/u) = (\mu x)/u = x/u$ . Cancelling  $x/u$ , we get  $\mu = \mathbf{1}$  which contradicts 1. The implication is proved.

2  $\Rightarrow$  3: Immediate.

3  $\Rightarrow$  4: By the conditions of this theorem,  $P_k \cdots P_1$  has an eigenvector  $y$  with eigenvalue  $\lambda$ . As any vector is greater than or equal to its image by the projector  $P_i$ , we have that  $\lambda \leq \mathbf{1}$ . Assume that  $\lambda = \mathbf{1}$ . Then the inequalities

$$P_k \cdots P_1 y \leq P_{k-1} \cdots P_1 y \leq \dots \leq y \quad (31)$$

turn into equalities, and  $y$  is a common vector of  $V_1, \dots, V_k$ , which contradicts 3.

4  $\Rightarrow$  1: Take  $x = y$ .  $\square$

To illustrate Theorem 16, consider the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & -\infty \\ 1 & 2 & -\infty & 1 \\ 0 & -1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 & 2 \\ 0 & 0 & 0 \\ -\infty & 0 & -1 \end{pmatrix}. \quad (32)$$

Let  $a^i$  and  $b^i$  denote the  $i$ -th column of  $A$  and  $B$ , respectively. For all vectors  $x = (x_1, \dots, x_n)$  and  $\beta > 0$ , we denote by  $\exp(\beta x)$  the vector of the same size with entries  $\exp(\beta x_j)$ . We define  $V_1$  (resp.  $V_2$ ) to be the subsemimodule of  $\mathbf{R}_{\max, m}^3$  generated by the vectors  $\exp(\beta a^i)$  for  $1 \leq i \leq 4$  (resp.  $\exp(\beta b^i)$  for  $1 \leq i \leq 3$ ). The discussions which follow are independent of the choice of the scaling parameter  $\beta > 0$ , which is adjusted to make Figure 1 readable (we took  $\beta = 2/3$ ). The two semimodules  $V_1, V_2$  and their generators are represented as follows at the left of the figure. Here, a non-zero vector  $w = (w_1, w_2, w_3) \in \mathbf{R}_{\max, m}^3$  is represented by the point of the two dimensional simplex which is the barycenter with weights  $w_j$  of the three vertices of this simplex. The generators  $a^i$  and  $b^i$  correspond to the bold dots. The semimodules  $V_1$  and  $V_2$  correspond to the two medium grey regions, together with the bold broken segments joining the generators to each of these regions.

Since the entries of  $x^0 := b^2 = \exp(\beta(2, 0, 0)) \in V_2$  are nonzero, the vector  $x^0$  is archimedean, and one can check, using the explicit formula of the projector (Theorem 5 of [5]), that  $x^0$  is an eigenvector of  $P_2 P_1$ . Indeed,

$$x^1 := P_1 x^0 = \exp(\beta(-1, 0, 0)^T) \quad (33)$$

and

$$x^2 := P_2 x^1 = \exp(\beta(-1, -3, -3)^T) = \exp(-3\beta)x^0 . \quad (34)$$

The halfspaces constructed in the proof of Theorem 16 are given by

$$\begin{aligned} H_1 &= \{u \mid x^0/u = x^1/u\} \\ &= \{u \mid \min(\exp(2\beta)/u_1, 1/u_2, 1/u_3) = \min(\exp(-\beta)/u_1, 1/u_2, 1/u_3)\} \\ &= \{u \mid \max(u_2, u_3) \geq \exp(\beta)u_1\} \end{aligned} \quad (35)$$

and

$$\begin{aligned} H_2 &= \{u \mid x^1/u = x^2/u\} \\ &= \{u \mid \min(\exp(-\beta)/u_1, 1/u_2, 1/u_3) = \min(\exp(-\beta)/u_1, \exp(-3\beta)/u_2, \exp(-3\beta)/u_3)\} \\ &= \{u \mid u_1 \geq \exp(2\beta) \max(u_2, u_3)\}. \end{aligned} \quad (36)$$

These two halfspaces are represented by the zones in light gray (right). The proof of Theorem 16 shows that their intersection is zero (meaning that it is reduced to the zero vector).

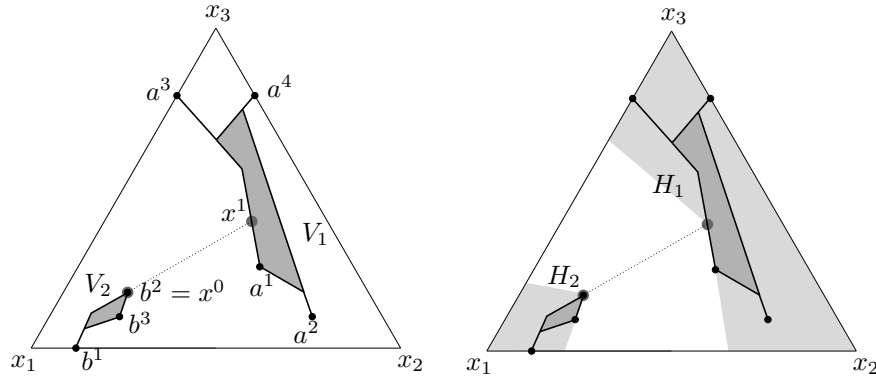


Figure 1: Two semimodules (left) with separating halfspaces (right)

## 4 Cyclic projectors and separation theorems in $\mathbf{R}_{\max, m}^n$

In  $\mathbf{R}_{\max, m}^n$ , it is natural to consider semimodules that are closed in the Euclidean topology. One can easily show that such semimodules are  $b$ -semimodules. Theorem 3.11 of [6] implies that projectors onto closed subsemimodules of  $\mathbf{R}_{\max, m}^n$  are continuous.

In order to relax the assumption concerning archimedean vectors in Theorem 16, we shall use some results from nonlinear spectral theory, that we next recall. By Brouwer's fixed point theorem, a continuous homogeneous operator  $x \mapsto Fx$  that maps  $\mathbf{R}_+^n$  to itself has a

nonzero eigenvector. This allows us to define the nonlinear spectral radius of  $F$ ,

$$\rho(F) = \max\{\lambda \in \mathbf{R}_+ \mid \exists x \in (\mathbf{R}_+^n) \setminus 0, Fx = \lambda x\} . \quad (37)$$

Suppose in addition that  $F$  is isotone. Then it can be checked that if  $Fx = \lambda x$ ,  $Fy = \mu y$ , and if  $x$  and  $y$  are comparable, then  $\lambda = \mu$ . It follows that the number of eigenvalues of  $F$  is bounded by the number of nonempty supports of vectors of  $\mathbf{R}_+^n$ , i.e. by  $2^n - 1$ . This implies in particular that the maximum is attained in (37). We shall need the following nonlinear generalization of the Collatz-Wielandt formula for the spectral radius of a nonnegative matrix.

**Theorem 17** (R.D. Nussbaum, Theorem 3.1 of [18]) *For any isotone, homogeneous, and continuous map  $F$  from  $\mathbf{R}_+^n$  to itself, we have:*

$$\rho(F) = \inf_{x \in (\mathbf{R}_+ \setminus \{0\})^n} \max_{1 \leq i \leq n} [F(x)]_i x_i^{-1}. \quad (38)$$

This result implies that the spectral radius of such operators is isotone: if  $F(x) \leq G(x)$  for any  $x \in \mathbf{R}_+^n$ , then  $\rho(F) \leq \rho(G)$ .

As the projectors on subsemimodules of  $\mathbf{R}_{\max, m}^n$  are isotone, homogeneous and continuous, so are their compositions, i.e. cyclic projectors. Consequently, we can apply Theorem 17 to them. Hence, if  $V'_i$ ,  $i = 1, \dots, k$  and  $V_i$ ,  $i = 1, \dots, k$  are closed semimodules in  $\mathbf{R}_{\max, m}^n$  and such that  $V'_i \subseteq V_i$ ,  $i = 1, \dots, k$ , then

$$\rho(P'_k \cdots P'_1) \leq \rho(P_k \cdots P_1), \quad (39)$$

as the projectors are isotone with respect to inclusion (7).

In the following theorem we characterize the spectrum of cyclic projectors in  $\mathbf{R}_{\max, m}^n$ .

**Theorem 18** *Let  $V_1, \dots, V_k$  be closed semimodules in  $\mathbf{R}_+^n$ . Then the Hilbert value  $d_H(V_1, \dots, V_k)$  is the spectral radius of  $P_k \cdots P_1$ . The spectrum of  $P_k \cdots P_1$  is the set of Hilbert values  $d_H(V_1^M, \dots, V_k^M)$ , where  $M$  ranges over all nonempty subsets of  $\{1, \dots, n\}$ .*

*Proof.* We first prove that the Hilbert value  $d_H(V_1, \dots, V_k)$  is the spectral radius of the cyclic projector, and hence an eigenvalue. We take  $k$  elementary subsemimodules spanned by  $x^i \in V_i$ ,  $i = 1, \dots, k$  and consider elementary projectors  $P'_i$  onto them. Observe that

$$\rho(P'_k \cdots P'_1) = d_H(x^1, \dots, x^k). \quad (40)$$

Denote by  $\bar{x}^0$  an eigenvector of  $P_k \cdots P_1$ , associated with the spectral radius, and let  $\bar{x}^i = P_i \cdots P_1 \bar{x}^0$ . Then

$$\rho(P_k \cdots P_1) = d_H(\bar{x}^1, \dots, \bar{x}^k). \quad (41)$$

By (39) it follows that  $\rho(P_k \cdots P_1) \geq \rho(P'_k \cdots P'_1)$ , that is,  $d_H(\bar{x}^1, \dots, \bar{x}^k) \geq d_H(x^1, \dots, x^k)$  for any  $x^1 \in V_1, \dots, x^k \in V^k$ . Thus, the Hilbert value of  $V_1, \dots, V_k$  is the spectral radius of  $P_k \cdots P_1$ .

Now consider  $d_H(V_1^M, \dots, V_k^M)$  for arbitrary  $M \subseteq \{1, \dots, n\}$ . Note that the semimodules  $V_1^M, \dots, V_k^M$  are closed, and denote by  $P_1^M, \dots, P_k^M$  the projectors onto these. It is easy to see that  $P_i^M(y) = P_i(y)$  for all  $i$  and all  $y$  with  $\text{supp}(y) \subseteq M$ . It follows that  $d_H(V_1^M, \dots, V_k^M)$  is the spectral radius of  $P_k^M \cdots P_1^M$  and also an eigenvalue of  $P_k \cdots P_1$ .

We have proved that any Hilbert value  $d_H(V_1^M, \dots, V_k^M)$  is an eigenvalue of  $P_k \cdots P_1$ . The converse statement follows from Theorem 13.  $\square$

The following three results refine Theorem 16.

**Theorem 19** *Suppose that  $V_1, \dots, V_k$  are closed archimedean subsemimodules of  $\mathbf{R}_{\max, m}^n$ . The following are equivalent:*

1. *there exist a positive vector  $x$  and a number  $\lambda < 1$  such that  $P_k \cdots P_1 x \leq \lambda x$ ;*
2. *there exist archimedean halfspaces  $H_i$  which contain  $V_i$  and are such that  $H_1 \cap \dots \cap H_k = \{\mathbf{0}\}$ ;*
3.  *$V_1 \cap \dots \cap V_k = \{\mathbf{0}\}$ ;*
4.  *$\rho(P_k \cdots P_1) < 1$ .*

*Proof.* The implications  $1 \Rightarrow 2$ ,  $2 \Rightarrow 3$ ,  $3 \Rightarrow 4$  are proved in Theorem 16. The implication  $4 \Rightarrow 1$  follows from Equation (38).  $\square$

**Proposition 20** *Suppose that  $V_i$ ,  $i = 1, \dots, k$  are closed semimodules in  $\mathbf{R}_{\max, m}^n$  with zero intersection. Then there exist closed archimedean semimodules  $V'_i$ ,  $i = 1, \dots, k$  with zero intersection and such that each  $V'_i$  contains  $V_i$ .*

*Proof.* In every semimodule  $V_i$ , we find a vector  $y^i$  with maximal support and such that  $\|y^i\| = \max(y_1^i, \dots, y_n^i) = 1$ . For all scalars  $\delta > 0$ , define

$$z^i(\delta) = y^i \oplus \delta \bigoplus_{j \notin \text{supp}(y^i)} e^j \quad (42)$$

and the semimodules

$$V_i(\delta) = \{x \mid x = v \oplus \lambda z^i, v \in V'_i, \lambda \in \mathbf{R}_+\}. \quad (43)$$

These semimodules are closed, as all arithmetical operations are continuous. We show that for  $\delta > 0$  small enough these semimodules have zero intersection. Assume by contradiction that for all  $\delta > 0$ , there exists a nonzero vector  $u(\delta)$  in the intersection  $V_1(\delta) \cap \cdots \cap V_k(\delta)$ . After normalizing  $u(\delta)$ , we may assume that  $\|u(\delta)\| = 1$ . For any  $i = 1, \dots, k$  and any  $\delta$  we have that

$$u(\delta) = v^i(\delta) \oplus \lambda_i(\delta)y^i \oplus \lambda_i(\delta)\delta \bigoplus_{j \notin \text{supp}(y^i)} e^j, \quad (44)$$

where  $v^i(\delta)$  is a vector from  $V_i'$  and  $\lambda_i(\delta)$  is a scalar. As  $\|u(\delta)\| = 1$  and  $\|y^i\| = 1$ , we have that  $\lambda_i(\delta) \leq 1$ . So, there exists a sequence  $(\delta_m)_{m \geq 1}$  converging to 0 such that for all  $1 \leq i \leq k$ ,  $\lambda_i(\delta_m)$  converges to a limit as  $m$  tends to infinity. Then  $w := \lim_{m \rightarrow \infty} u(\delta_m) = \lim_{m \rightarrow \infty} v_i(\delta_m) \oplus \lambda_i(\delta_m)y^i$  for all  $i$ . As  $V_i$  are closed,  $w$  belongs to  $V_i$  at all  $i$ . Since  $\|w\| = 1$ ,  $w$  is not equal to  $\mathbf{0}$ , which is a contradiction.  $\square$

The following is an immediate corollary of Theorem 19 and Proposition 20.

**Theorem 21** (Separation theorem) *If  $V_i$ ,  $i = 1, \dots, k$  are closed semimodules with zero intersection, then there exist archimedean halfspaces  $H_i$ ,  $i = 1, \dots, k$ , which contain the corresponding semimodules  $V_i$  and have zero intersection.*

The following separation theorem for two closed semimodules is a corollary of Theorem 21.

**Theorem 22** *If  $U$  and  $V$  are two closed semimodules with zero intersection, then there exists a closed halfspace  $H_U$ , which contains  $U$  and has zero intersection with  $V$ , and there exists a closed halfspace  $H_V$ , which contains  $V$  and has zero intersection with  $U$ .*

As a consequence of Theorem 21, we further deduce a separation theorem for convex subsets of  $\mathbf{R}_{\max, m}^n$ . We recall here some definitions from idempotent convex geometry, see e.g. [12]. A subset  $C \subset \mathbf{R}_{\max, m}^n$  is *convex* if  $\lambda u \oplus \mu v \in C$ , for all  $u, v \in C$  and  $\lambda, \mu \in \mathbf{R}_{\max, m}$  such that  $\lambda \oplus \mu = 1$ .

The *recession cone* of a convex set  $C$ ,  $\text{rec}(C)$ , is the set of vectors  $u$  such that  $v \oplus \lambda u \in C$  for all  $\lambda \in \mathbf{R}_{\max, m}$ , where  $v$  is an arbitrary vector of  $C$ . As shown in Prop. 2.6 of [12], if  $C$  is closed, the recession cone is independent of the choice of  $v$ . Observe that when  $C$  is compact, its recession cone is zero.

A set  $H^{\text{aff}}$  given by

$$H^{\text{aff}} = \{x \mid u/x \wedge \alpha \geq v/x \wedge \gamma\} \quad (45)$$

with  $u, v \in \mathbf{R}_{\max, m}^n$ ,  $u \leq v$ ,  $\alpha, \gamma \in \mathbf{R}_{\max, m}$ ,  $\alpha \leq \gamma$ , will be called (*idempotent*) *affine halfspace*. It is called *archimedean*, if  $u, v, \alpha$  and  $\gamma$  are positive.

For a convex set  $C \subset \mathbf{R}_{\max, m}^n$  define  $V(C) \subset \mathbf{R}_{\max, m}^{n+1}$  to be the semimodule of vectors of the form  $(x_1\lambda, \dots, x_n\lambda, \lambda)$  with  $x = (x_1, \dots, x_n) \in C$  and  $\lambda \in \mathbf{R}_{\max, m}$ .

**Theorem 23** (Separation of convex sets) *Let  $C_1, \dots, C_k$  be closed convex subsets of  $\mathbf{R}_{\max, m}^n$  with empty intersection, and assume that the intersection of the recession cones of  $C_1, \dots, C_k$  is zero. Then, there exist affine archimedean halfspaces  $H_1^{\text{aff}}, \dots, H_k^{\text{aff}}$  which contain the corresponding convex sets  $C_i$ ,  $i = 1, \dots, k$  and have empty intersection.*

*Proof.* From Prop. 2.16 of [12], we know that the closure of  $V(C_i)$ ,  $\overline{V(C_i)}$ , is equal to  $V(C_i) \cup (\text{rec}(C_i) \times \{0\})$ . Hence, the assumptions imply that the intersection of  $\overline{V(C_1)}, \dots, \overline{V(C_k)}$  is zero. By Theorem 21, we can find archimedean halfspaces  $H_i \supset \overline{V(C_i)}$  with zero intersection. Every  $H_i$  can be written as

$$H_i = \{(x_1, \dots, x_n, \mu) \mid u^i/x \wedge \alpha^i/\mu \geq v^i/x \wedge \gamma^i/\mu\} \cup \{0\} \quad (46)$$

with  $u^i \leq v^i$  and  $\alpha^i \leq \gamma^i$ , understanding that  $x := (x_1, \dots, x_n)$ . Observe that for all  $x \in C_i$ ,  $(x, 1) \in V(C_i) \subset H_i$ . We deduce that the affine archimedean halfspace

$$H_i^{\text{aff}} = \{x \mid u^i/x \wedge \alpha^i \geq v^i/x \wedge \gamma^i\} \quad (47)$$

contains  $C_i$ . Since the intersection of the halfspaces  $H_i$  is zero, the intersection of the affine halfspaces  $H_i^{\text{aff}}$  must be empty.  $\square$

In convex analysis, one can find an analogous separation theorem for several compact convex sets, see [11], pages 39-40.

We now deduce an idempotent analogue of the classical Helly's Theorem. As observed by S. Gaubert and F. Meunier [13], there is another proof of this theorem, which is based on the direct idempotent analogue of Radon's argument (see [11]).

**Theorem 24** (Helly's Theorem) *Suppose that  $V_i$ ,  $i = 1, \dots, m$  is a collection of  $m \geq n$  semimodules in  $\mathbf{R}_{\max, m}^n$ . If each  $n$  semimodules intersect nontrivially, then the whole collection has a nontrivial intersection.*

*Proof.* It suffices to consider the case where the semimodules  $V_i$  are all closed. Indeed, the assumption implies that for all  $j := (j_1, \dots, j_n) \in \{1, \dots, m\}^n$ , we can choose a non-zero element  $z_j$  in the intersection  $V_{j_1} \cap \dots \cap V_{j_n}$ . Let  $V_i'$  denote the semimodule generated by the elements  $z_j$  that belong to  $V_i$ . The collection of semimodules  $V_i'$ ,  $i = 1, \dots, m$  still has the property that each  $n$  semimodules intersect nontrivially. Moreover,  $V_i'$  is closed, because it is finitely generated (see e.g. Lemma 2.20 of [12] or Corollary 27 of [4]). Hence, if the

conclusion of the theorem holds for closed semimodules, we deduce that the whole collection  $V'_i, i = 1, \dots, m$  has a nontrivial intersection, and since  $V_i \supset V'_i$ , the conclusion of the theorem also holds without any closure assumption.

In the discussions that follow, the semimodules  $V_i$  are all closed. We argue by contradiction, assuming that the whole collection has zero intersection. By Theorem 20, we can also assume that the semimodules  $V_i$  are archimedean. For some number  $k < m$  every  $k$  semimodules intersect nontrivially, but there are  $k + 1$  semimodules, say  $V_1, \dots, V_{k+1}$ , which have zero intersection. By Theorem 19, there exists a positive vector  $y = y^0$  and a scalar  $\lambda < 1$  such that

$$P_{k+1} \cdots P_1 y \leq \lambda y. \quad (48)$$

For all  $i$  we denote  $y^i = P_i \cdots P_1 y^0$ , where projectors are indexed modulo  $(k + 1)$ . By the homogeneity and isotonicity of projectors, we have that

$$P_{l+k+1} \cdots P_{l+1} y^l \leq \lambda y^l \quad (49)$$

for all  $l = 1, \dots$ . Consider the vectors

$$z^l = P_{l+k} \cdots P_{l+1} y^l \quad (50)$$

for  $l = 1, \dots, k + 1$ . Since each  $k$  semimodules intersect nontrivially, the vector  $z^l$  must have at least one coordinate equal to that of  $y^l$ , for otherwise  $y^l$  would satisfy the first condition of Theorem 19, giving a contradiction. As  $k \geq n$ , there are at least two numbers  $l$  and at least one number  $i$  such that  $z^l$  has the same  $i$ th coordinate as  $y^l$ . If we take the smallest of these two  $l$  numbers, then

$$(P_{l+k+1} \cdots P_{l+1} y^l)_i = y_i^l. \quad (51)$$

But this contradicts (49). Hence any  $k + 1$  semimodules intersect nontrivially, which is again a contradiction. The theorem is proved.  $\square$

There is also an affine version of this theorem.

**Theorem 25** *Suppose that  $C_i, i = 1, \dots, m$  is a collection of  $m \geq n + 1$  convex subsets of  $\mathbf{R}_{\max, m}^n$ . If each  $n + 1$  of these convex sets have a nonempty intersection, then the whole collection has a nonempty intersection.*

*Proof.* Consider the semimodules  $V(C_1), \dots, V(C_m)$  defined above, and apply Theorem 24 to them.  $\square$

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