An Exploration of the Fibonacci Sequence

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Abstract

The famous mathematical sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots,\ m,\ n,\ m + n, \ldots$, known as the Fibonacci sequence, has been discovered in many places such as nature, art, and even music. In this paper, we will discuss the history of the Fibonacci sequence and the man that the Fibonacci sequence was named after, Leonardo Fibonacci. We will also discuss the problem solved by Leonardo Fibonacci that made this sequence famous, the golden ratio and how it involves Fibonacci numbers, the occurrence of the Fibonacci numbers in nature, equiangular spirals, and the use of Fibonacci numbers in music. Finally, we will discuss some of the mathematics behind the sequence and other neat properties of the Fibonacci sequence.

1 Introduction

The famous mathematical sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots,\ m,\ n,\ m + n, \ldots$, known as the Fibonacci sequence, is a natural wonder to not only the mathematical world, but also fields such as botany, biology, and even music. The Fibonacci numbers are found in various plants in nature and even used by composers to provide a beautiful harmony in their music. The ratio of two successive Fibonacci numbers is equal to a proportion known as the golden ratio, which we will discuss later, and is equal to proportions found on the human anatomy or in the growth pattern of a chambered mollusk shell. Overall, the Fibonacci sequence is fascinating because of the numerous places its numbers can be found and utilized. In this paper we will discuss some of the history behind the Fibonacci sequence and the life of the man it was named after, Leonardo Fibonacci. We will also discuss the rabbit problem, the golden ratio, equiangular spirals, the occurrence of Fibonacci numbers in nature, the use of Fibonacci numbers in music, and some mathematical proofs concerning the Fibonacci sequence.

2 A History of the Fibonacci Sequence

2.1 Leonardo Fibonacci

Leonardo Fibonacci was born in Pisa, Italy in 1175 and was known as Leonardo of Pisa. When he was a young boy, he traveled with his father to the north coast of Africa where his father worked as a customs official in a warehouse in Algeria. While there, he is believed to have studied from Muslim schoolmasters and learned the Hindu-Arabic numeral system. While still a young boy, he traveled with his father to Egypt, Syria, Greece, Sicily and southern France. As he visited each of these places, he observed the merchants' methods of calculating and was convinced that the Hindu-Arabic numeral system was more efficient than those he had examined. He later wrote *Liber Abaci* in 1202 under the name Leonardo Fibonacci, which was derived from *filius Bonacci* meaning "son of Bonacci".[1] The book was based on the arithmetic and algebra that Fibonacci had learned during his travels and influenced the change from Roman numerals to Hindu-Arabic numerals in Europe.[10] In this same book, Fibonacci introduced a problem known as the "rabbit problem," in which the solution contained the sequence that would later be called the Fibonacci sequence. Some of Fibonacci's other works include *Practica Geometriae* (1220), and *Liber Quadratorum* (1225), and in 1228, he revised his work *Liber Abaci*. He also wrote three or four minor works, one being a commentary of the Tenth Book of Euclid of Alexandria. Leonardo Fibonacci is believed to have died in about 1250, but his work did not. Although a slow change, the use of the Hindu-Arabic numeral system spread throughout Europe and is still used today. Today, there is a statue of Leonardo Fibonacci in Pisa, Italy to commemorate a man that "brought light to the science of mathematics." [1]

2.2 What's in a Name?

The Fibonacci sequence was unofficially recognized by the ancient Egyptians and their Greek students.[3] In 1878, a French mathematician named Edouard Lucas studied the sequence that he would later name the Fibonacci sequence because of his fascination of the "rabbit problem" from which it originated.[5] By using the same pattern of finding a term as the Fibonacci sequence by adding the two before it, he used 1 and 3 as his two initial terms instead of 1 and 1 as in the Fibonacci sequence and created his own sequence known as the Lucas numbers.

3 The Rabbit Problem

"How many pairs of rabbits can be produced from a single pair in a year if every month each pair of begets a new pair, which, from the second month on, becomes productive?" The solution to this problem made the sequence famous. As stated, start with one pair of young adult rabbits. They will have babies monthly. Their babies mature two months after birth. Then these rabbits will have babies monthly. In the first month, we start with one rabbit. By the second month, they have had a pair of rabbits, making the total two. By the next month, the first pair of rabbits has another pair of rabbits, making the total three. By the beginning of April, the first pair will have had another pair of rabbits and the first pair that it produced will have matured and had their own pair of rabbits, making the total five. This continues until the end of the year when the total number of pairs of rabbits equals Fibonacci's solution of 377 pairs. In figure 1, we see the rabbits born through June with their babies that they would have had throughout the rest of the year. Although Fibonacci's solution to the problem is mathematically correct, looking at

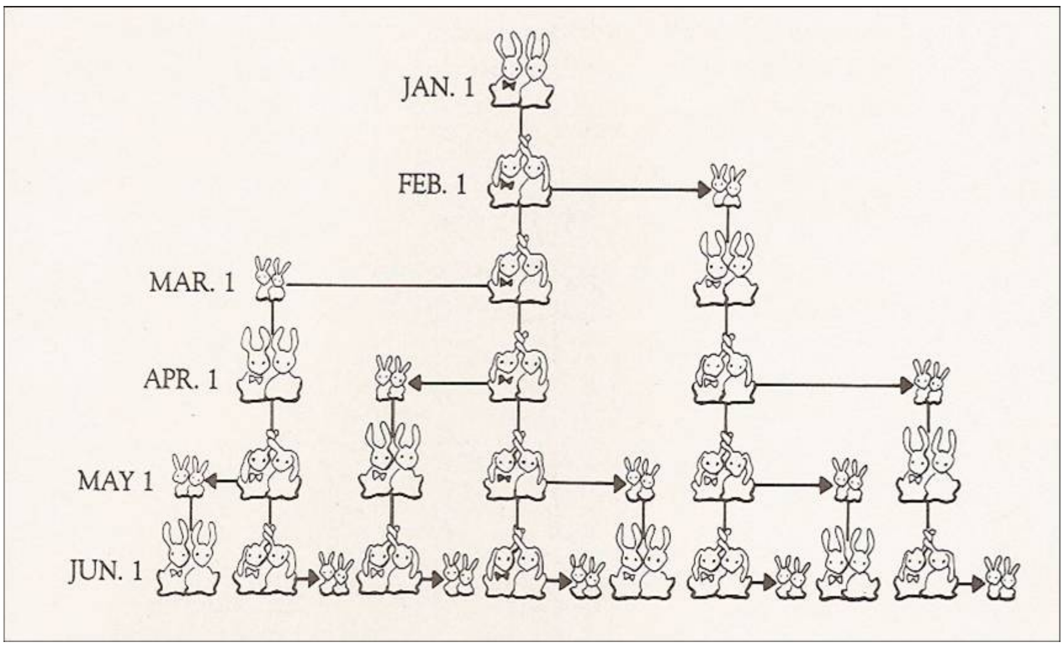

Figure 1: Rabbits [1]

the problem from a biologists point-of-view and considering that there was not any outside interference, we could conclude that rabbits do not actually breed this way.

4 The Golden Ratio and the Divine Proportion

The golden ratio, also known as the divine proportion, has been around since the beginning of time. From the proportions found on the human body to the spiral of a shell, this fascinating number is present in many natural occurrences.

4.1 Finding the Golden Ratio

The golden ratio can be found by the ratio of two successive Fibonacci numbers. Observe that

$$\frac{F_n}{F_{n+1}} = \frac{F_{n+1}}{F_n + F_{n+1}}$$

By examining the proportions of the term numbers, we see

$$\frac{n}{n+1} = \frac{n+1}{n+1+n}$$

$$\frac{n}{n+1} = \frac{n+1}{2n+1}$$

$$n(2n+1) = (n+1)(n+1)$$

$$2n^2 + n = n^2 + 2n + 1$$

$$n^2 - n - 1 = 0$$

By using the quadratic formula, we find the roots to be

$$n = \frac{1+\sqrt{5}}{2}$$

and

$$n = \frac{1-\sqrt{5}}{2}.[1]$$

These two numbers later became known as ϕ and ψ, respectively.

4.2 ϕ and ψ

Looking at the ratios of the Fibonacci numbers, where the larger number is on top and its smaller adjacent term on bottom, we see that

$$\frac{F_2}{F_1} = 1/1 = 1.00$$

$$\frac{F_3}{F_2} = 2/1 = 2.00$$

$$\frac{F_4}{F_3} = 3/2 = 1.50$$

$$\frac{F_5}{F_6} = 5/3 = 1.67$$

$$\frac{F_6}{F_7} = 8/5 = 1.60$$

$$\vdots$$

$$\frac{F_{14}}{F_{13}} = 377/233 = 1.618026$$

$$\frac{F_{15}}{F_{14}} = 610/377 = 1.618027$$

$$\vdots$$

The limit of these ratios converges to the approximation 1.61803. Also, the limit of the inverses of the ratios, the smaller Fibonacci number on top and its larger adjacent term on bottom, converges to the approximation .61803. These numbers are approximations of

$$\phi = \frac{1+\sqrt{5}}{2}$$

and

$$\psi = \frac{1-\sqrt{5}}{2},$$

respectively.

5 The Equiangular Spirals

The equiangular spiral, sometimes known as *spira mirabilis* meaning "wonderful spiral," has been a common occurrence in the natural world for millions of years. In 1638, French mathematician René Descartes named the curve the equiangular spiral because the angle at which a radius vector cuts the curve at any point is constant. Since the radius increases in geometrical progression as its polar angle increases in arithmetical progression, it has been called the geometrical spiral. English astronomer, Edmond Halley called this curve a proportional spiral since the lengths of the segments cut off from a fixed radius by successive turns of the curve were in continued proportion. Jacob Bernoulli (1654-1705), who studied logarithms and exponential functions, was so impressed with this curve that he had it put on his tombstone. He found the curve to be logarithmic, so he called it a logarithmic spiral.[2] Regardless of what it may be called, this curve remains fascinating and can be associated with the Fibonacci sequence through the proportion of its curves.

5.1 Construction of a Golden Spiral

A golden spiral can be drawn by using golden rectangles. First, you start with two squares with side lengths 1. Now, a square with side lengths 2 can be connected to the two unit squares. Then, on one of the sides with a length of $1 + 2$, a square with side lengths 3 can be drawn. If this is continued, the length of the sides of the next square drawn is the sum of the previous two drawn. By drawing a quarter circle in each of the squares using the side as a radius and connecting each square's quarter circle at the end of the next larger square's quarter circle, a golden spiral will be formed. The squares also form rectangles which are called Fibonacci rectangles because the ratios of the sides are the ratios of successive Fibonacci numbers. Observe the golden spiral in figure 2.

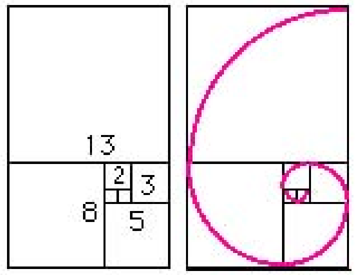

Figure 2: Construction of a Golden Spiral[12]

5.2 Spirals in Nature

As stated before, the equiangular spiral can be found in nature. It has been found in the spirals of leaves, shell shapes, horns of wild sheep, spider webs and even galaxies. A more specific example is the chambered nautilus. A chamber nautilus is a type of mollusk whos shell grows by the law of biological growth. This law deals with the growth of plants, animals, or even any part of them, and says that the growth is an exponential law. The shell of nautilus, which can be seen in figure 3, grows as the nautilus grows. As the nautilus grows, it needs a new chamber on its shell to grow into. Each chamber that grows onto the open end of the shell is a similar shape as the other chambers, but bigger. As the shell gets larger, the radii get larger, but the angles of intersection of the radii and the outer shell remain the same. Thus, the shell of a nautilus grows into an equiangular spiral.[2]

6 Fibonacci Sequence in Nature

Equiangular spirals are not the only case of the Fibonacci numbers occurring in nature. The Fibonacci numbers can also be found in the ancestry of a male bee, the spiraling scales of a pineapple, the spiraling seeds of a sunflower or coneflower, and the spiraling bracts of a pinecone. We will now examine these examples.

6.1 Genealogy of a Drone Bee

A drone, male, bee hatches from an unfertilized egg while a female bee hatches from a fertilized egg. Thus a male bee only has one parent, a female, and a female bee has two

Figure 3: Nautilus' Shell[6]

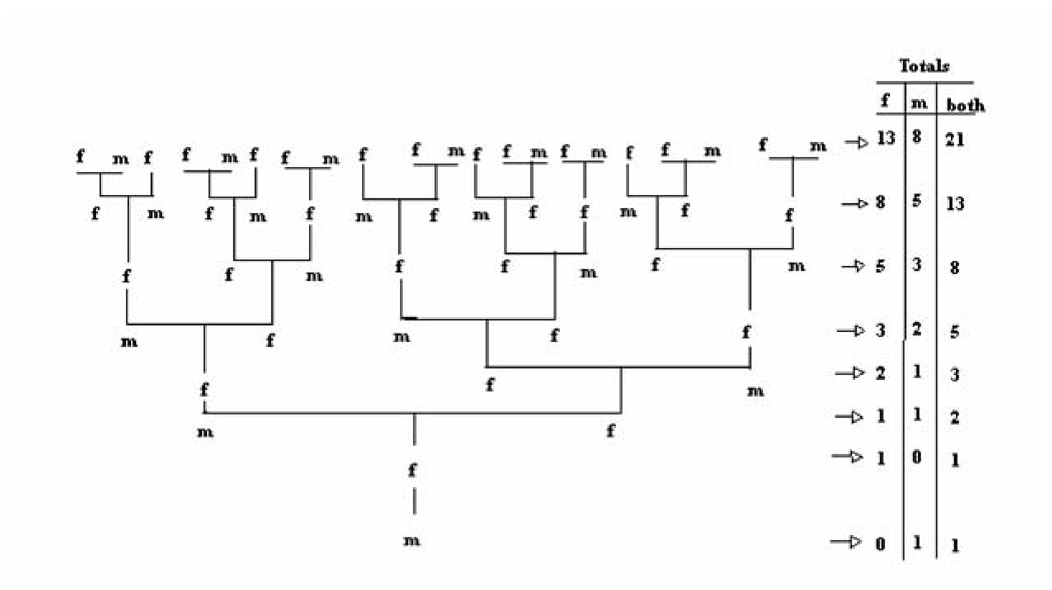

Figure 4: Male Bee's Genealogy

parents, a male and a female. When we trace the ancestry of a male bee, the Fibonacci sequence is formed. The ancestry of a male bee can be seen in figure 4. If we add up the number of male bees in each generation starting with the first, we can see that the Fibonacci sequence starts with the third generation. Now, if we add the number of female bees in each generation starting with the first, we can see that the Fibonacci sequence starts in the second generation. And finally, if we add the total number of bees in each generation starting with the first, we see that the Fibonacci sequence starts with the first generation. We can observe in the table below the total number of male bees, female bees, and the overall total of bees in each generation.

males	1	0	1	1	2	3	5	8
females	0	1	1	2	3	5	8	13
both	1	1	2	3	5	8	13	21

6.2 Spirals in Plants

6.2.1 A Pineapple's Spiraling Scales

A pineapple generally has three spirals that can be found on it. The sets {3,5,8}, {5,8,13}, and {8,13,21} can be found according to the size of the pineapple. A pineapple has hexagon-shaped scales making up its outer layer. Each scale can be found in three different diagonals on a pineapple. Consecutive sets of spirals run opposite of each other. (i.e., 5 would run opposite of 8). The higher the number of spirals, the steeper that spiral will be. Since consecutive Fibonacci numbers run opposite of each other and must each have a different steepness, no spirals moving opposite of each other will have the same spiral.[5] An example of a pineapple with a {3,5,8} set of spirals is shown in figure 5.

6.2.2 Sunflowers and Coneflowers

The spirals found in these flowers' seeds usually have 34 spirals going one way and 55 going the other. Giant sunflowers have 55 spirals going one way and 89 going the other. Some other sunflowers have been reported to have 89 and 144, or even 144 and 233. All of these numbers are consecutive Fibonacci numbers. There have even been reports of sunflowers having double Fibonacci numbers. For example, instead of 34 and 55, they would have 68 and 110. Observe that the coneflower in figure 6 has 34 spirals going to the left (green) and 55 spirals going to the right (blue).

6.2.3 Pinecones

A pinecone is another plant where spirals can be found. The bracts on a pinecone spiral in two different directions. The spirals found are always two consecutive Fibonacci numbers,

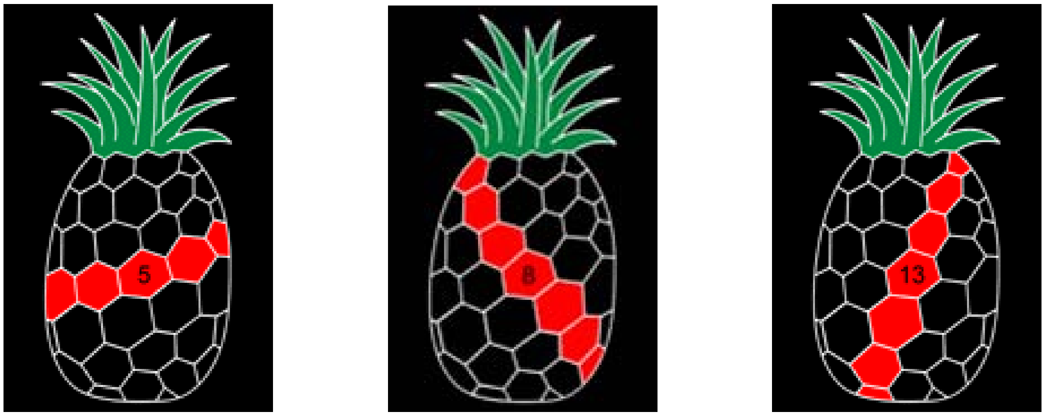

Figure 5: Pineapple's Spiraling Scales [4]

Figure 6: Coneflower (left) [6] and Sunflower (right) [11]

such as 3 and 5, 5 and 8, or even 8 and 13, according to the size of the pinecone. Like a pineapple, the two spirals run opposite ways and one is generally steeper than the other. In figure 7, an example of a pinecone with a set of 8 and 13 spirals can be seen.

7 Fibonacci Numbers in Music

The beauty of the Fibonacci numbers is more evident when found in the works of art, the sounds of music, or even the architecture of many buildings. The ratios formed by the Fibonacci numbers provide pleasing sights or sounds, which may be the reason they are used in paintings and compositions.

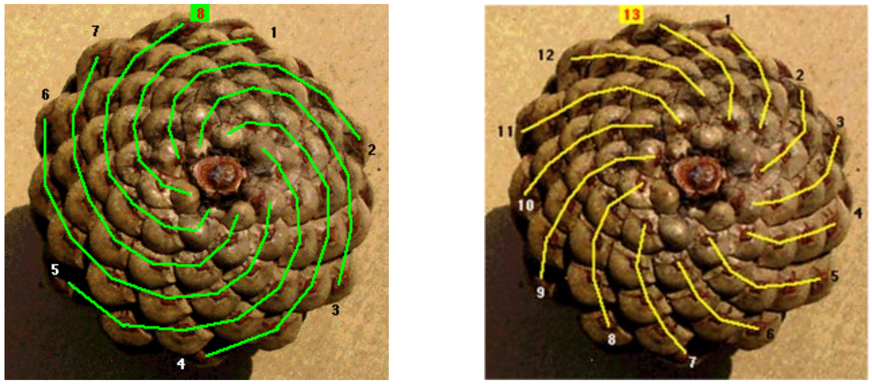

Figure 7: Pinecone's Spiraling Bracts [6]

7.1 Music and the Fibonacci Numbers

For a more pleasing sound, composers would divide musical time into periods based on a proportion. By doing this, they mark the beginnings and endings of themes in their pieces. On a piano, an octave contains 5 black keys, 8 white keys, and 13 total keys, with the black keys alternating with 2 or 3 at a time. These five numbers are all Fibonacci numbers. In Western music, there are three different scales: the chromatic scale, the diatonic scale, and the pentatonic scale. The chromatic scale uses the 13 total keys found in an octave. The diatonic scale uses the 8 white keys found in the octave, and the pentatonic scale uses any five consecutive black keys.[1]

7.1.1 The Major Sixth and Minor Sixth

It is no surprise that a good harmony sounds pleasant to your ears. When using the Major sixth and the Minor sixth, the ratio of the keys used reduces to Fibonacci numbers. With the Major sixth, the keys C and A are used. The C gets 264 vibrations per second while the A gets 440 vibrations per second. When put in a ratio, these two numbers form a proportion of 3 and 5, which are both Fibonacci numbers. The Minor sixth shares this similar characteristic. The keys E and C are used when in Minor sixth. The E gets 330 vibrations per second while the C gets 528 vibrations per second. When these two rates are put into a ratio, they are proportional to the ratio of 5 and 8, which are both Fibonacci numbers. Similar ratios can also be found with any other six-key interval.[1]

8 Mathematics with the Fibonacci Numbers

The Fibonacci sequence, $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$, is found by taking the two initial terms, $F_1 = F_2 = 1$, and adding them together to get the next term, $F_3 = 2$. To find the

F_n term, where n is a natural number, take the sum of the previous two terms. Observe that the nth term can be found with the formula $F_n = F_{n-2} + F_{n-1}$.

8.1 Binet's Formula

Another way to find the nth term of the Fibonacci sequence is to use a more explicit formula known as Binet's formula. Credited to Jacques Philippe Marie Binet, a French mathematician of the 18th century, the formula

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

can be used to find the nth term of the Fibonacci sequence, where F_n represents the nth Fibonacci term. Notice that this equation can also be written as

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}.$$

8.2 Properties of the Fibonacci Sequence

There are many properties within the Fibonacci sequence for which can be proven through a number of proofs. For instance, the sums of Fibonacci numbers and the sums of the squares of Fibonacci numbers are two properties of the Fibonacci sequence.

8.2.1 The Sum of the First n Fibonacci Numbers

Let's begin by proving that the sum of the first n Fibonacci numbers is equal to one less than the $(n+2)$ term, $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$. First, we will calculate the sum of the first n Fibonacci numbers. Let the first two terms be the initial terms $F_1 = F_2 = 1$. Since $F_3 = F_2 + F_1$, it follows that $F_1 = F_3 - F_2$. Likewise,

$$\begin{aligned}
F_4 &= F_3 + F_2 \\
F_5 &= F_4 + F_3 \\
F_6 &= F_5 + F_4 \\
&\vdots \\
F_{n+1} &= F_n + F_{n-1} \\
F_{n+2} &= F_{n+1} + F_n,
\end{aligned}$$

so

$$F_2 = F_4 - F_3$$

$$F_3 = F_5 - F_4$$
$$F_4 = F_6 - F_5$$
$$\vdots$$
$$F_{n-1} = F_{n+1} - F_n$$
$$F_n = F_{n+2} - F_{n+1}.$$

When we add the Fibonacci terms, they must equal the sum of these equations. Observe that

$$F_1 + F_2 + F_3 + \cdots + F_{n-1} + F_n = \\ (F_3 - F_2) + (F_4 - F_3) + (F_5 - F_4) + (F_6 - F_5) + \cdots + (F_{n+1} - F_n) + (F_{n+2} - F_{n+1})$$

$$F_1 + F_2 + F_3 + \cdots + F_{n-1} + F_n = -F_2 + F_{n+2}$$

Since $F_2 = 1$, it follows that

$$F_1 + F_2 + F_3 + \cdots + F_{n-1} + F_n = F_{n+2} - 1$$

Thus, the sum of the first n Fibonacci numbers is equal to one less than the $n+2$nd term.[8]

8.2.2 The Sum of the Squares of the First n Fibonacci Numbers

Now we will prove that the sums of the squares of the first n Fibonacci numbers is equal to the product of the nth and $n+1$st Fibonacci terms. First, we will find the sum of the squares of the first n Fibonacci numbers. Let

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_{n-1}^2 + F_n^2$$

represent the sum of the squares of the first n Fibonacci numbers. We know that for every nth term of the Fibonacci sequence, F_n,

$$F_{n+1} = F_n + F_{n-1}$$

so we know that

$$F_n = F_{n+1} - F_{n-1}.$$

This means that we can write

$$F_n^2 = F_n(F_{n+1} - F_{n-1}).$$

It follows that

$$F_n^2 = F_nF_{n+1} - F_nF_{n-1}.$$

For the first n terms of the Fibonacci sequence, we observe that

$$F_1^2 = F_1F_2$$

$$F_2^2 = F_2F_3 - F_2F_1$$
$$F_3^2 = F_3F_4 - F_3F_2$$
$$\vdots$$
$$F_{n-1}^2 = F_{n-1}F_n - F_{n-1}F_{n-2}$$
$$F_n^2 = F_nF_{n+1} - F_nF_{n-1}.$$

Now, we can add the squares of the Fibonacci numbers and their equations. Observe that

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_{n-1}^2 + F_n^2 =$$
$$F_1F_2 + (F_2F_3 - F_2F_1) + (F_3F_4 - F_3F_2) + \cdots + (F_{n-1}F_n - F_{n-1}F_{n-2}) + (F_nF_{n+1} - F_nF_{n-1}),$$

which can then be reduced to

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_{n-1}^2 + F_n^2 = F_nF_{n+1}.$$

Thus, the sum of the squares of the first n Fibonacci numbers is equal to the product of the nth and $n+1$st terms.[8]

8.2.3 Other Properties

Other dividing properties of the Fibonacci numbers are as follows:

1. A Fibonacci number is even if and only if its term number is divisible by 3. Note that the sequence is as follows: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$. Since the two initial terms are 1 and 1, and the sum of two odd numbers is even, the third term, 2, would be even. Since the sum of an even and an odd number is odd, the 4th and 5th terms are going to be odd since the sums both include 2 and an odd number (1 or 3). This continues throughout the sequence. So let's look at every third term of the sequence. Observe that the sequence of every third term of the Fibonacci sequence is as follows: $2, 8, 34, 144, 610, 2584, \ldots$. Notice that all of these numbers are divisible by 2. Therefore, every third Fibonacci term is an even number.

2. A Fibonacci number is divisible by 3 if and only if its term number is divisible by 4. So let's look at every fourth term of the sequence. Observe that the sequence of every fourth term of the Fibonacci sequence is as follows: $3, 21, 144, 987, 6765, 46368, \ldots$. Notice that all of these numbers are divisible by 3. Therefore, every fourth Fibonacci term is divisible by 3.

3. A Fibonacci number is divisible by 4 if and only if its term number is divisible by 6. So let's look at every sixth term of the sequence. Observe that the sequence of every sixth term of the Fibonacci sequence is as follows: $8, 144, 2584, 46368, 832040, 14930352, \ldots$. Notice that all of these numbers are divisible by 4. Therefore, every sixth Fibonacci term is divisible by 4.

4. A Fibonacci number is divisible by 5 if and only if its term number is divisible by 5. So let's look at every fifth term of the sequence. Observe that the sequence of every fifth term of the Fibonacci sequence is as follows: $5, 55, 610, 6765, 75025, 832040, \ldots$. Notice that all of these numbers are divisible by 5. Therefore, every fifth Fibonacci term is divisible by 5.

5. A Fibonacci number is divisible by 7 if and only if its term number is divisible by 8. So let's look at every eighth term of the sequence. Observe that the sequence of every eighth term of the Fibonacci sequence is as follows: $21, 987, 46368, 2178309, 102334155, 4807526976, \ldots$. Notice that all of these numbers are divisible by 7. Therefore, every eighth Fibonacci term is divisible by 7.

9 Conclusion

In conclusion, we have discussed Leonardo Fibonacci's history and how the Fibonacci sequence came to be defined. We found the golden ratio and discussed its importance to the Fibonacci sequence, which led us to Binet's formula. We observed examples of the occurrence of the Fibonacci numbers in nature and music. We also defined what an equiangular spiral is and discussed how to construct one using the Fibonacci numbers. And finally, we discussed a few mathematical proofs involving the Fibonacci numbers. Overall, this sequence is fascinating because of the different places its numbers naturally appear.

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