Math 364
Spring, 2004 Name:
Final Exam

Show all work and explain your reasoning. Answer all questions. Start all problems on the top of the front of a new page of your blue book. Each short answer problem can be completed on one page, and work can carry over to the back of the page.

Good luck!

1. **Definitions.** (3 points each = 24 points) Complete each statement with the required definition.
   (a) Let A_α for $\alpha \in \Delta$ be an indexed family of sets. The intersection $\bigcap_{\alpha\in\Delta} A_\alpha$ is ____________________.
   (b) For two integers a and b the least common multiple of a and b is denoted _____ and is defined to be ____________________________.
   (c) Let A and B be sets. Then $A - B$ is defined to be ____________________________.
   (d) $\mathcal{A}$ is a partition of A iff ____________________________.
   (e) Let R be a relation from A to B. The domain of R is denoted _____________ and is defined to be ____________________________.
   (f) f is a function from A to B iff ____________________________.
   (g) A function $f : A \to B$ is one-to-one iff ____________________________.
   (h) Let R be a relation on a set A. The canonical map is a function from _____________ to _____________ given by ____________________________.

2. (3 points) State the Principle of Mathematical Induction.

3. (3 points) State the Well-ordering principle.

4. **True / False.** (4 points each = 16 points) State if the following are true or false. If true, provide a brief proof. If false, provide a counterexample.
   (a) Let A and B be sets. Then $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
   (b) If $A \subseteq B$ and $B \subseteq C$ then $A = C$ and $B = C$.
   (c) Let the relation V on the natural numbers given by $a\ V\ b$ iff a is relatively prime to b. V is an equivalence relation.
   (d) Let R be the relation on the natural numbers given by $a\ R\ b$ iff $a - b \geq 0$. Then R is an equivalence relation.

5. **Examples.** (3 points each = 18 points) Provide examples of each of the items below **or** explain why such an example is impossible. Be sure to explain why your example is appropriate.

   (a) A true conditional statement with a false inverse.

   (b) A proposition which is a tautology.

   (c) A relation from $A = \{1, 2, 3\}$ to $B = \{4, 5, 6\}$ which is not a function.

   (d) A relation from $A = \{1, 2, 3\}$ to $B = \{4, 5, 6\}$ which is reflexive and transitive, but not symmetric.

   (e) A function from $A = \{1, 2, 3\}$ to $B = \{4, 5, 6\}$ which is onto but not one-to-one.

   (f) Let $X = \{1, 2, 3, 4, \ldots, 20\}$. Give an example of a family $\mathcal{A}$ of pairwise disjoint subsets of X such that $\mathcal{A}$ has 4 elements and $\bigcap_{A\in\mathcal{A}} A = \{1\}$ and $\bigcup_{A\in\mathcal{A}} A = X$.

6. (4 points) Let $X = \{1, 2, 3, 4, 5\}$ and let $\mathcal{A} = \{\{1, 2\}, \{3\}, \{4, 5\}\}$ be a partition on X. List all ordered pairs that occur in the equivalence relation corresponding to this partition.

Prove each of the following.
Be sure that your explanations are clear and complete.

7. (4 points) Prove or disprove: Let A, B, and X be sets. The $A \subseteq B \iff X - B \subseteq X - A$

8. (4 points) Prove or disprove: Let X and Y be sets. $X = (X - Y) \cup (X \cap Y)$

9. (6 points) Prove or disprove: For each natural number n, $5^n - 2^n$ is divisible by 3.

10. (5 points) Prove or disprove: If a, b, and c are integers such that a and b are relatively prime and $a \mid bc$ then $a \mid c$.

11. (5 points) Prove or disprove: Let R be an equivalence relation on a non-empty set A. Then $xRy \iff x/R = y/R$.

12. (6 points) Prove or disprove: Every natural number $n > 1$ is either prime or is the product of prime factors.

13. (6 points) Prove or disprove: If p is prime then $\sqrt{p}$ is irrational.

14. (6 points) We say that a relation R on a set A is antisymmetric if, for all $x, y \in A$, if $x\ R\ y$ and $y\ R\ x$ then $x = y$. Show that if R is antisymmetric then $x\ R\ y$ and $x \neq y$ implies $y \not R x$

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This list is **NOT** exhaustive, but covers many of the ideas we have discussed in class. Ideas which do not appear on this list may appear on the exam. To prepare, you should also review homework and exam problems.

1. Know the following definitions:
   - paradox
   - tautology and contradiction
   - conditional proposition, converse, inverse, contrapositive
   - a divides b
   - even and odd
   - direct proof, proof by contradiction, proof by contraposition
   - existential proofs
   - $\exists!$ proofs
   - GCD
   - LCM
   - union (two sets and family of sets) and intersection (two sets and family of sets)
   - set difference
   - power set
   - complement of a set
   - mutually disjoint
   - Partition
   - PMI
   - Division algorithm and Euclidean algorithm (know ideas and how to use)
   - Cartesian product
   - relation
   - domain and range
   - inverse of a relation and composite of two relations
   - equivalence relation and equivalence class, $[x] = x/R$ and A/R
   - function, domain, range, codomain, image, preimage
   - characteristic, canonical, identity, projection, and inclusion functions
   - one-to-one function, onto functions, and bijections
2. Be able to prove the following theorems:
   (a) There is an infinite number of primes.
   (b) $\sqrt{p}$ is irrational where p is prime.
   (c) Every natural number $n > 1$ is either prime or is the product of prime factors.