Math 364
Spring, 2005 Name:
Final Exam

Show all work and explain your reasoning. Answer all questions. Good Luck!!!

1. **Definitions.** Complete each statement with the required definition.
   (a) For two integers a and b the least common multiple, m, of a and b is defined to be
   (b) Let A be a set. Then the power set of A, $\mathcal{P}(A)$, is defined to be
   (c) Let A_α for $\alpha \in \Delta$ be an indexed family of sets. The union $\bigcup_{\alpha\in\Delta} A_\alpha$ is

2. **True / False.** State if the following are true or false. If true, provide a brief proof. If false, provide a counterexample.
   (a) For all real numbers x, there exists a real number y such that $xy = 1$.
   (b) $C \subseteq A \cap B$ iff $C \subseteq A$ and $C \subseteq B$
   (c) If $A \cup C \subseteq B \cup C$ then $A \subseteq B$
   (d) Let R be the relation on the natural numbers given by $a\ R\ b$ iff $a \geq b$. Then R is an equivalence relation.
   (e) Let $X = \{1, 2, 3\}$ and consider the relation on $\mathcal{P}(x)$ given by $A \sharp B$ iff $A \cap B = \emptyset$. Then $\sharp$ is an equivalence relation.

3. **Examples.** Provide examples of each of the items below **or** explain why such an example is impossible. Be sure to explain why your example is appropriate.
   (a) A false conditional statement with a true contrapositive.
   (b) A true conditional statement with a false inverse.
   (c) A relation R on $\{1, 2, 3\}$ such that R is symmetric, but not transitive or reflexive.
   (d) A relation from $A = \{1, 2, 3\}$ to $B = \{4, 5, 6\}$ which is reflexive and transitive, but not symmetric.
   (e) Let $X = \{1, 2, 3, 4, \ldots, 20\}$. Give an example of a family $\mathcal{A}$ of subsets of X such that $\mathcal{A}$ has 4 elements and $\bigcap_{A\in\mathcal{A}} A = \{1\}$ and $\bigcup_{A\in\mathcal{A}} A = X$.

Prove each of the following.
Be sure that your explanations are clear and complete.

4. Prove or disprove: Let A, B, C, and D be sets with $C \subseteq A$ and $D \subseteq B$. If A and B are disjoint then C and D are disjoint.

5. Prove or disprove: Let $\mathcal{A}$ be a family of pairwise disjoint sets. If $\mathcal{B} \subseteq \mathcal{A}$ then $\mathcal{B}$ is a family of pairwise disjoint sets.

6. Prove or disprove: Let X and Y be sets. $X = (X - Y) \cup (X \cap Y)$

7. Prove or disprove: The sum of the cubes of any three consecutive natural numbers is a multiple of 9.

8. Prove or disprove: For every natural number n, $n^4 - 6n^3 + 23n^2 - 18n$ is divisible by 24.

9. Prove or disprove: If a, b, and c are integers such that a and b are relatively prime and $a \mid bc$ then $a \mid c$.

10. Prove or disprove: $LCM(a, b) = ab$ if and only if a divides b.

11. Prove or disprove: Every natural number $n > 1$ is either prime or is the product of prime factors.

12. Prove or disprove: If p is prime then $\sqrt{p}$ is irrational.

13. For any integers a and b, $a < b$ if and only if $b = a + n$ for some natural number n. Prove or disprove: If $a < b$ and $c < d$ then $a + c < b + d$.

14. We say that a relation R on a set A is antisymmetric if, for all $x, y \in A$, if $x \ R \ y$ and $y \ R \ x$ then $x = y$. Show that if R is antisymmetric then $x \ R \ y$ and $x \neq y$ implies $y \not R x$.