Math 364 - Set Theory Problems
Fall 2008

Definition 24 *A set is a specified collection of objects.*

Example 7 *The set of integers between* -1 *and* 4*, including the endpoints, is* $\{-1, 0, 1, 2, 3, 4\}$.

Definition 25 *If* A *is a set and* x *is an object that belongs to* A *then* x *is an element of* A*, denoted* $x \in A$*. If* x *is not an object that belongs to* A*, we say that* x *is not an element of* A *(or is not a member of* A*), denoted* $x \notin A$*.*

79. List all of the elements in the following sets:
    (a) The set of natural numbers strictly less than 6.
    (b) The set of integers whose square is less than 17.
    (c) The set of prime numbers less than 100.
    (d) The set of rational numbers strictly between 0 and 1.

Definition 26 *If* A *and* B *are sets,* B *is a subset of* A*, denoted* $B \subseteq A$ *if every member of* B *is a member of* A*.*

80. True or False, and explain your reasoning. $\mathbb{N}$ is the set of natural numbers, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{Z}$ is the set of integers, and $\mathbb{R}$ is the set of real numbers.
    (a) $\mathbb{N} \subseteq \mathbb{Q}$
    (b) $\mathbb{Z} \subseteq \mathbb{N}$
    (c) $\mathbb{Q} \subseteq \mathbb{Z}$
    (d) $\mathbb{N} \subseteq \mathbb{R}$
    (e) $\mathbb{R} \subseteq \mathbb{Q}$
    (f) $(6, 9] \subseteq [6, 10)$
    (g) $[7, 10] \subseteq \mathbb{R}$

Definition 27 *Sets* A *and* B *are equal, denoted* $A = B$*, if* $A \subseteq B$ *and* $B \subseteq A$*.*

Definition 28 *The set with no members is called the* <u>*empty set*</u> *and denoted* $\emptyset$*.*

81. True or False and explain your reasoning.
    (a) $\emptyset \subseteq \mathbb{N}$
    (b) $\emptyset \in \mathbb{N}$
    (c) $\emptyset \in \{\emptyset, \{\emptyset\}\}$
    (d) $\emptyset \subseteq \{\emptyset, \{\emptyset\}\}$
    (e) $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$
    (f) $\{\emptyset, \{\emptyset\}\} \subseteq \{\{\emptyset, \{\emptyset\}\}\}$

82. True or False and explain your reasoning.
    (a) For every set A, $\emptyset \subseteq A$
    (b) For every set A, $\emptyset \in A$

83. Give an example, if there is one, of sets A, B, and C such that the following are true. If there is no example, state such. Explain your reasoning on all problems.
    (a) $A \subseteq B, B \nsubseteq C$ and $A \subseteq C$.
    (b) $A \subseteq B, B \subseteq C$, and $C \subseteq A$.
    (c) $A \nsubseteq B, B \nsubseteq C$ and $A \subseteq C$.
    (d) $A \subseteq B, B \nsubseteq C$ and $A \nsubseteq C$.

84. ($*$) Let A be a set. Then $\emptyset \subseteq A$ and $A \subseteq A$.

85. ($*$) Let A, B and C be sets. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Definition 29 *A subset* $A \subseteq B$ *is a* <u>*proper subset*</u> *if* $A \subseteq B$ *and* $A \neq B$*. This is denoted by* $A \subset B$*.*

86. List all of the subsets of the following sets. Which ones are proper?
    (a) $\emptyset$
    (b) $\{1\}$
    (c) $\{1, 2\}$
    (d) $\{\{\emptyset\}\}$
    (e) $\{\emptyset, \{\emptyset\}\}$

Definition 30 *The* <u>*power set*</u> *of a set* A *is the set of all (proper and not proper) subsets of* A*. This is denoted* $\mathcal{P}(A)$*.*

87. Give an example, if there is one, of each of the following. If there is no example, state such. Explain your reasoning on all problems.

   (a) A set A such that $\mathcal{P}(A)$ has 64 elements.
   (b) Sets A and B such that $A \subseteq B$ and $\mathcal{P}(B) \subseteq \mathcal{P}(A)$.
   (c) A set A such that $\mathcal{P}(A) = \emptyset$
   (d) A set A such that $\mathcal{P}(A) = \{\emptyset\}$
   (e) Sets A, B, and C such that $A \subseteq B$, $B \subseteq C$ and $\mathcal{P}(A) \subseteq \mathcal{P}(C)$.

88. True or False and explain your reasoning.

   (a) $\emptyset \in \mathcal{P}(\{\emptyset, \{\emptyset\}\})$
   (b) $\{\emptyset\} \in \mathcal{P}(\{\emptyset, \{\emptyset\}\})$
   (c) $\{\{\emptyset\}\} \in \mathcal{P}(\{\emptyset, \{\emptyset\}\})$
   (d) $\emptyset \subseteq \mathcal{P}(\{\emptyset, \{\emptyset\}\})$
   (e) $\{\emptyset\} \subseteq \mathcal{P}(\{\emptyset, \{\emptyset\}\})$
   (f) $\{\{\emptyset\}\} \subseteq \mathcal{P}(\{\emptyset, \{\emptyset\}\})$

89. ($*$) Prove or disprove: Let n be a natural number and A be a set containing n elements. The number of elements in $\mathcal{P}(A)$ is 2^n.

   For all of the following, A, B, and C are sets.

90. True or False and explain your reasoning.

   (a) The empty set is a proper subset of every set.
   (b) If A is a proper subset of $\emptyset$, then $A = \{17\}$.
   (c) If $A \subseteq B$ then $A = B$.
   (d) If $A = B$ then $A \subseteq B$.
   (e) Since $\emptyset$ is a member of $\{\emptyset\}$, $\emptyset = \{\emptyset\}$.
   (f) There is a set that is a member of every set.

Definition 31 *If A and B are sets, then the union of A and B is the set of all objects that belong to A or belong to B, denoted $A \cup B$. In other words,*

$$A \cup B = \{x \mid x \in A \textit{ or } x \in B\}$$

Definition 32 *If A and B are sets, then the intersection of A and B is the set of all objects that belong to both A and B, denoted $A \cap B$. In other words,*

$$A \cap B = \{x \mid x \in A \textit{ and } x \in B\}$$

91. Prove or disprove: $\emptyset \cap A = \emptyset$ and $\emptyset \cup A = A$

92. $(*)$ Prove or disprove: $A \cap B \subseteq A$

93. $(*)$ Prove or disprove: $A \subseteq A \cup B$

94. Prove or disprove: $A \cup B = B \cup A$ and $A \cap B = B \cap A$

95. Prove or disprove: $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$

96. Prove or disprove: $A \cup A = A = A \cap A$

97. $(*)$ Prove or disprove: If $A \subseteq B$, then $A \cup C \subseteq B \cup C$ and $A \cap C \subset B \cap C$.

Definition 33 *Let A and B be sets. Then the complement of A relative to B is the set $\{x \in B \mid x \notin A\}$, denoted $B - A$.*

Definition 34 *If for a certain problem, all of the sets being considered are subsets of a given set U then U is called a universal set.*

Definition 35 *If U is the universal set and $A \subseteq U$ then the complement of A relative to U is denoted A' and is $U - A = A' = \{x \in U \mid x \notin A\}$*

98. Prove or disprove: $(A')' = A$

99. Prove or disprove: $(A \cup B)' = A' \cap B'$

100. Prove or disprove: $(A \cap B)' = A' \cup B'$

101. $(*)$ Prove or disprove: $A - B = A \cap B'$

102. $(*)$ Prove or disprove: $A \subseteq B$ if and only if $B' \subseteq A'$

103. Prove or disprove: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

104. Prove or disprove: $(A \cap B) \cup C = A \cap (B \cup C)$

105. Prove or disprove: $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

Definition 36 *Two sets A and B are disjoint if $A \cap B = \emptyset$.*

106. $(*)$ Prove or disprove: $A \cap B$ and $A - B$ are disjoint.

107. Prove or disprove: $A = (A \cap B) \cup (A - B)$.

108. $(*)$ Prove or disprove: $A - (A \cap B') = A \cap B$

109. Prove or disprove: If A and B are sets such that $A \cup B = A \cap B$ then $A \cap B' = \emptyset$.

110. $(*)$ Prove or disprove: If A and B are sets such that $(A \cup B)' = A' \cup B'$ then $A = B$.

111. (∗) Prove or disprove: Let A, B and C be sets such that $A \cup B \neq A \cap C$. Then A is not a subset of C or B is not a subset of A.

Definition 37 *A set of sets is called a* <u>*family*</u> *or* <u>*collection*</u> *of sets.*

Example 8

$$\mathcal{A} = \{\{1,2,3\},\{3,4,5\},\{3,6\},\{2,3,6,7,9,10\}\}$$

is a family consisting of four sets.

112. Consider the family $\mathcal{A} = \{\{1,2,3\},\{3,4,5\},\{3,6\},\{2,3,6,7,9,10\}\}$. Answer the following true/false questions, with reasons.

   (a) $\{3,4,5\} \in \mathcal{A}$

   (b) $3 \in \mathcal{A}$

   (c) $\{3,4,5\} \subseteq \mathcal{A}$

   (d) $\{\{3,4,5\}\} \subseteq \mathcal{A}$

Definition 38 *Let $\mathcal{A}$ be an family of sets. The* <u>*union*</u> *over $\mathcal{A}$ is*

$$\bigcup_{A \in \mathcal{A}} A = \{x \mid x \in A \textit{ for some set } A \in \mathcal{A}\}$$

Definition 39 *Let $\mathcal{A}$ be an family of sets. The* <u>*intersection*</u> *over $\mathcal{A}$ is*

$$\bigcap_{A \in \mathcal{A}} A = \{x \mid x \in A \textit{ for every set } A \in \mathcal{A}\}$$

113. Consider the family $\mathcal{A} = \{\{1,2,3\},\{3,4,5\},\{3,6\},\{2,3,6,7,9,10\}\}$. Answer the following, with explanations:

   (a) Find $\displaystyle\bigcup_{A \in \mathcal{A}} A$

   (b) Find $\displaystyle\bigcap_{A \in \mathcal{A}} A$

114. Let $A_n = \{1,2,3,\ldots,n\}$ and let $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$. Answer the following, with explanations:

   (a) Find $\displaystyle\bigcup_{A_n \in \mathcal{A}} A_n$

   (b) Find $\displaystyle\bigcap_{A_n \in \mathcal{A}} A_n$

115. Consider the family $\mathcal{A} = \{[a, \infty) \mid a \in \mathbb{R}\}$. Answer the following, with explanations:

(a) Find $\bigcup_{A \in \mathcal{A}} A$

(b) Find $\bigcap_{A \in \mathcal{A}} A$

116. For each natural number n, let $A_n = \left(0, \frac{1}{n}\right)$ and let $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$. Answer the following, with explanations:

(a) Find $\bigcup_{A \in \mathcal{A}} A$

(b) Find $\bigcap_{A \in \mathcal{A}} A$

117. For each $n \in \mathbb{Z}$, let $C_n = [n, n+1)$ and let $\mathcal{C} = \{C_n \mid n \in \mathbb{Z}\}$. Answer the following, with explanations:

(a) Find $\bigcup_{A \in \mathcal{A}} A$

(b) Find $\bigcap_{A \in \mathcal{A}} A$

118. For each $n \in \mathbb{Z}$, let $A_n = (n, n+1)$ and let $\mathcal{A} = \{A_n \mid n \in \mathbb{Z}\}$. Answer the following, with explanations:

(a) Find $\bigcup_{A \in \mathcal{A}} A$

(b) Find $\bigcap_{A \in \mathcal{A}} A$

119. ($*$) Prove or disprove: For every set B in a family $\mathcal{A}$ of sets, $\bigcap_{A \in \mathcal{A}} A \subseteq B$

120. Prove or disprove: For every set B in a family $\mathcal{A}$ of sets, $B \subseteq \bigcup_{A \in \mathcal{A}} A$

121. ($*$) Prove or disprove: If the family $\mathcal{A}$ contains at least one set, then $\bigcap_{A \in \mathcal{A}} A \subseteq \bigcup_{A \in \mathcal{A}} A$.