Math 364 - Relation Problems
Spring 2009

Definition 40 *Let S be a set. Let $a, b \in S$. An ordered pair is a double (a, b) where a is the first term of the ordered pair and b is the second term of (a, b). A relation R on S is a set of ordered pairs where the elements of the ordered pairs come from S.*

Fact 3 *The set of first terms is sometimes called the domain of the relation. The set of second terms is sometimes called the range of the relation.*

Example 9 *Let $S = \{2, 4, 6, 8, 10\}$. Then a relation on S could be defined by*

$$R = \{(2, 2), (4, 4), (6, 6), (6, 8), (6, 10), (8, 6), (8, 8), (8, 10), (10, 6), (10, 8), (10, 10)\}.$$

This relation can be described as two numbers from S are related if they have the same number of divisors. It can also be written as $2R2$ meaning that 2 is related to 2 under the relation R.

122. Let $S = \{1, 2\}$. List all of the possible relations on S.

123. Let $A = \{1, 2, 3, 4, 5\}$. Write all elements of the following relations:

    (a) $R = \{(a, b) \in A \times A \mid a \text{ divides } b\}$

    (b) $E = \{(a, b) \in A \times A \mid a + b \text{ is even }\}$

    (c) $U = \{(a, b) \in A \times A \mid a \neq b\}$

Definition 41 *Let A and B be sets. Then $A \cup B$ is another set, and we can define a relation on $A \cup B$ called the Cartesian product of A and B, denoted $A \times B$, by $A \times B = \{(a, b) \mid a \in A$ and $b \in B\}$. In other words, this relation is the set of all ordered pairs where the first term is an element of A and the second term is an element of B.*

124. ($*$) Let $A = \{1, 2, 3\}$ and $B = \{2, 5\}$. List all elements of $A \times B$. What is the domain of this relation? What is the range of this relation?

125. Let $A = \{1\}$, $B = \{2\}$, and $C = \{3\}$. Show that $A \times (B \times C) \neq (A \times B) \times C$.

126. ($*$) Let A have n elements and B have m elements. Prove or disprove: $A \times B$ has mn elements.

127. Let A and B be non-empty sets. Prove that $A \times B = B \times A$ if and only if $A = B$.

Definition 42 *Let S be a nonempty set and R a relation on S. Then*

(a) The relation R is reflexive on S if for each $x \in S$, $(x, x) \in R$.

(b) The relation R is symmetric on S if $(y, x) \in R$ whenever $(x, y) \in R$.

(c) The relation R is transitive on S if whenever $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

128. Let $S = \{3, 5\}$. Is the relation $R = \{(3, 5)\}$ reflexive? symmetric? transitive? Explain.

129. Let $S = \{1, 2, 3\}$. Find a relation on S that is reflexive and symmetric, but not transitive.

130. Let $S = \{1, 2, 3\}$. Find a relation on S that is reflexive and transitive, but not symmetric.

131. Let $S = \{1, 2, 3\}$. Find a relation on S that is symmetric and transitive, but not reflexive.

Definition 43 *An equivalence relation on a set S is a relation that is reflexive, symmetric, and transitive.*

132. Let $S = \{1, 2, 3\}$. Find a relation on S that is an equivalence relation.

133. Let $S = \{1, 2, 3\}$. Find a relation on S that is not reflexive, not symmetric, and not transitive.

134. Let $S = \mathbb{R}$. For any two real numbers a and b, define $a \simeq b$ iff $a^2 = b^2$. Prove that $\simeq$ is an equivalence relation on $\mathbb{R}$. List all of the elements that are equivalent to -7.

135. Let $S = \mathbb{R} \times \mathbb{R}$. Define $(a, b) \simeq (c, d)$ iff $a^2 + b^2 = c^2 + d^2$. Prove that $\simeq$ is an equivalence relation on $\mathbb{R} \times \mathbb{R}$. List all of the elements that are equivalent to $(0, 0)$.

136. Let $\mathcal{A} = \{\{1, 2\}, \{3, 4\}, \{5, 6, 7\}, \{8\}\}$.

    (a) List all elements of $\bigcup \mathcal{A}$

    (b) Let $R = \{(x, y) \in \bigcup \mathcal{A} \times \bigcup \mathcal{A} \mid x \text{ and } y \text{ belong to the same member of } \mathcal{A}\}$. Prove that R is an equivalence relation on $\bigcup \mathcal{A}$.

    (c) List all elements that are related to 5.

Definition 44 *Let S be a non-empty set and R an equivalence relation on S. The set of all elements equivalent (i.e. related to) an element $a \in S$ are the equivalence class of a, denoted $[a]$.*