Math 244
Fall, 2004
Review of Integrals

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1. Evaluate the following integrals:

(a) $\int \frac{dx}{9-x^2}$

$$x = 3\sin\theta \qquad dx = 3\cos\theta d\theta$$

$$= \int \frac{3\cos\theta d\theta}{9-9\sin^2\theta}$$
$$= \int \frac{3\cos\theta d\theta}{9\cos^2\theta}$$
$$= \frac{1}{3}\int \frac{d\theta}{\cos\theta}$$
$$= \frac{1}{3}\int \sec\theta d\theta$$
$$= \frac{1}{3}\ln \mid \sec\theta + \tan\theta \mid + C$$

(b) $\int \frac{dx}{\sqrt{9-x^2}}$

$$= \int \frac{1}{3}\cdot \frac{dx}{\sqrt{1-\frac{x^2}{9}}}$$
$$= \frac{1}{3}\int \frac{dx}{\sqrt{1-\left(\frac{x}{3}\right)^2}}$$
$$u = \frac{x}{3} \qquad du = \frac{1}{3}dx$$
$$= \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}}$$
$$= \frac{1}{3}\sin^{-1}\frac{x}{3} + C$$

(c) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

$$u = \sqrt{x} \qquad du = \frac{1}{2\sqrt{x}}dx$$
$$= \int 2\cos u du$$
$$= 2\sin u + C$$
$$= 2\sin\sqrt{x} + C$$

(d) $\int \frac{dx}{\sqrt{-2x-x^2}}$

$$= \int \frac{dx}{\sqrt{(-x^2+2x)}}$$
$$= \int \frac{dx}{\sqrt{-(x^2+2x+1-1)}}$$
$$= \int \frac{dx}{\sqrt{-(x+1)^2+1}}$$
$$= \int \frac{dx}{\sqrt{1-(x+1)^2}}$$
$$u = x+1 \qquad du = dx$$
$$= \int \frac{1}{\sqrt{1-u^2}} du$$
$$= \sin^{-1} u + C$$
$$= \sin^{-1}(x+1) + C$$

(e) $\int \frac{2-\cos x+\sin x}{\sin^2 x} dx$

$$= \int \frac{2}{\sin^2 x} - \frac{\cos x}{\sin^2 x} + \frac{\sin x}{\sin^2 x} dx$$
$$= \int 2\csc^2 x - \cot x \csc x + \csc x dx$$
$$= -2\cot x + \csc x - \ln|\csc x + \cot x| + C$$

(f) $\int \theta \cos(2\theta+1) d\theta$

Use integration by parts.

$$u = \theta \qquad dv = \cos(2\theta+1)d\theta$$
$$du = d\theta \qquad v = \frac{1}{2}\sin(2\theta+1)$$

$$\int \theta\cos(2\theta+1)d\theta = \frac{1}{2}\theta\sin(2\theta+1) - \frac{1}{2}\int \sin(2\theta+1)d\theta$$
$$= \frac{1}{2}\theta\sin(2\theta+1) + \frac{1}{4}\cos(2\theta+1) + C$$

(g) $\int \frac{x^3}{x^2-2x+1} dx$

Use polynomial long division to rewrite this as:

$$= \int x + 2 + \frac{3x-2}{x^2-2x+1} dx$$
$$= \frac{x^2}{2} + 2x + \int \frac{3x-2}{(x-1)^2}$$

$$= \frac{x^2}{2} + 2x + \int \frac{A}{x-1} + \frac{B}{(x-1)^2} dx$$

Solving for the coefficients:

$$A(x-1) + B = 3x - 2$$

When $x = 1$:

$$B = 3(1) - 2 = 1$$

$$Ax - A + 1 = 3x - 2$$

$$A = 3$$

So the integral is:

$$= \frac{x^2}{2} + 2x + \int \frac{3}{x-1} + \frac{1}{(x-1)^2} dx$$

$$= \frac{x^2}{2} + 2x + 3\ln\mid x-1\mid - \frac{1}{x-1} + C$$

(h) $\int \frac{dx}{\sqrt{1+\sqrt{x}}}$

$$u = 1 + \sqrt{x} \qquad u - 1 = \sqrt{x} \qquad du = \frac{dx}{2\sqrt{x}}$$

$$2\sqrt{x}du = dx \qquad 2(u-1)du = dx$$

so the integral is;

$$= \int \frac{2(u-1)du}{\sqrt{u}}$$

$$= 2\int \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}} du$$

$$= 2\left(\frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u}\right) + C$$

$$= \frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}} - 2\sqrt{1+\sqrt{x}} + C$$

(i) $\int \frac{2\sin\sqrt{x}dx}{\sqrt{x}\sec\sqrt{x}}$

$$u = \sqrt{x} \qquad du = \frac{1}{2\sqrt{x}}dx$$

$$= \int \frac{4\sin u du}{\sec u}$$

$$= 4\int \sin u \cos u du$$

$$w = \sin u \qquad dw = \cos u du$$

$$= 4\int w dw$$

$$= 4\frac{w^2}{2} + C$$

$$= 4\frac{\sin^2\sqrt{x}}{2} + C$$
$$= 2\sin^2\sqrt{x} + C$$

(j) $\displaystyle\int \frac{\sin 2x dx}{(1+\cos 2x)^2}$

$$u = 1 + \cos 2x \qquad du = -2\sin 2x dx$$
$$-\frac{1}{2}dx = \sin 2x dx$$
$$= -\frac{1}{2}\int \frac{du}{u^2}$$
$$= -\frac{1}{2}\left(-\frac{1}{u}\right) + C$$
$$= \frac{\frac{1}{2}}{1+\cos 2x} + C$$

(k) $\displaystyle\int \frac{dy}{y^2 - 2y + 2}$

$$= \int \frac{dy}{y^2 - 2y + 1 - 1 + 2}$$
$$= \int \frac{dy}{(y^2 - 2y + 1) + 1}$$
$$= \int \frac{dy}{(y-1)^2 + 1}$$
$$u = y - 1 \qquad du = dy$$
$$= \int \frac{du}{u^2 + 1}$$
$$= \tan^{-1} u + C$$
$$= \tan^{-1}(y-1) + C$$

(l) $\displaystyle\int \ln\sqrt{x-1}dx$

$$= \int \frac{1}{2}\ln(x-1)dx$$
$$= \frac{1}{2}\int \ln(x-1)dx$$
$$u = \ln(x-1) \qquad dv = dx$$
$$du = \frac{1}{x-1}dx \qquad v = x$$
$$= x\ln(x-1) - \int x\frac{1}{x-1}dx$$
$$= x\ln(x-1) - \int \frac{x-1+1}{x-1}dx$$

$$= x\ln(x-1) - \int 1 + \frac{1}{x-1}dx$$
$$= x\ln(x-1) - (x + \ln \mid x-1 \mid) + C$$
$$= x\ln(x-1) - x - \ln \mid x-1 \mid + C$$

(m) $\int \frac{x+1}{x^2(x^2+1)}dx$

$$= \int \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}dx$$

Solving for the coefficients:

$$Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 = x+1$$

If $x = 0$:

$$B = 1$$
$$Ax^3 + Ax + x^2 + 1 + Cx^3 + Dx^2 = x+1$$
$$(A+C)x^3 + (D+1)x^2 + Ax + 1 = x+1$$
$$A = 1$$

So we have

$$(1+C)x^3 + (D+1)x^2 + 1x + 1 = x+1$$
$$D + 1 = 0 \Rightarrow D = -1$$
$$1 + C = 0 \Rightarrow C = -1$$

So we have:

$$= \int \frac{1}{x} + \frac{1}{x^2} + \frac{-x-1}{x^2+1}dx$$
$$= \ln \mid x \mid - \frac{1}{x} + \int \frac{-x}{x^2+1}dx - \int \frac{1}{x^2+1}dx$$
$$= \ln \mid x \mid - \frac{1}{x} - \frac{1}{2}\ln(x^2+1) - \tan^{-1} x + C$$

(n) $\int x^3 e^{x^2} dx$

$$= \int x^2 x e^{x^2} dx$$

$$u = x^2 \qquad dv = xe^{x^2}dx$$
$$du = 2xdx \qquad v = \frac{1}{2}e^{x^2}$$
$$= \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2}dx$$
$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C$$

(o) $\int \sin^2 x dx$

$$= \int \frac{1 - \cos 2x}{2} dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

(p) $\int \frac{\cos(\sin^{-1} x)}{\sqrt{1 - x^2}} dx$

$$u = \sin^{-1} x \qquad du = \frac{1}{\sqrt{1 - x^2}} dx$$

$$= \int \cos u du$$

$$= \sin(\sin^{-1} x) + C$$

$$= x + C$$

(q) $\int \frac{e^t dt}{1 + e^t}$

$$u = 1 + e^t \qquad du = e^t dt$$

$$= \int \frac{1}{u} du$$

$$= \ln \mid u \mid + C$$

$$= \ln \mid 1 + e^t \mid + C$$

(r) $\int \frac{\cot v dv}{\ln(\sin v)}$

$$u = \ln(\sin v) \qquad du = \frac{1}{\sin v} \cdot \cos v dv = \frac{\cos v}{\sin v} dv = \cot v dv$$

$$= \int \frac{1}{u} du$$

$$= \ln \mid u \mid + C$$

$$= \ln |\ln \sin v| + C$$

(s) $\int (27)^{3\theta+1} d\theta$

$$u = 3\theta + 1 \qquad du = 3d\theta$$

$$= \int \frac{1}{3} (27)^u$$

$$= \frac{1}{3} \frac{1}{\ln 27} 27^u + C$$

$$= \frac{1}{3 \ln 27} 27^{3\theta+1} + C$$

(t) $\int e^x \cos(2x)dx$

$$u = \cos(2x) \qquad dv = e^x dx$$

$$du = -2\sin(2x)dx \qquad v = e^x$$

$$\int e^x \cos(2x)dx = e^x \cos(2x) + 2\int e^x \sin(2x)dx$$

$$u = \sin(2x) \qquad dv = e^x dx$$

$$du = 2\cos(2x)dx \qquad v = e^x$$

$$\int e^x \cos(2x)dx = e^x \cos(2x) + 2\left(e^x \sin(2x) - 2\int e^x \cos(2x)dx\right)$$

$$\int e^x \cos(2x)dx = e^x \cos(2x) + 2e^x \sin(2x) - 4\int e^x \cos(2x)dx$$

$$5\int e^x \cos(2x)dx = e^x \cos(2x) + 2e^x \sin(2x) + C$$

$$\int e^x \cos(2x)dx = \frac{1}{5}e^x \cos(2x) + \frac{2}{5}e^x \sin(2x) + C$$

(u) $\int \frac{dx}{x^2 - 3x + 2}$

$$= \int \frac{dx}{(x-2)(x-1)}$$

$$= \int \frac{A}{x-2} + \frac{B}{x-1}dx$$

Solving for the coefficients we have

$$A(x-1) + B(x-2) = 1$$

If $x = 1$

$$B(-1) = 1$$

$$B = -1$$

If $x = 2$

$$A = 1$$

So the integral is:

$$= \int \frac{1}{x-2} + \frac{-1}{x-1}dx$$

$$= \ln \mid x-2 \mid - \ln \mid x-1 \mid + C$$

$$= \ln\left|\frac{x-2}{x-1}\right| + C$$

(v) $\displaystyle\int \frac{dx}{(x^2-1)^{\frac{3}{2}}}$

$$x = \sec\theta \qquad\qquad dx = \sec\theta\tan\theta d\theta$$

$$\begin{aligned}
&= \int \frac{\sec\theta\tan\theta d\theta}{(\sec^2\theta - 1)^{\frac{3}{2}}} \\
&= \int \frac{\sec\theta\tan\theta d\theta}{(\tan^2\theta)^{\frac{3}{2}}} \\
&= \int \frac{\sec\theta\tan\theta d\theta}{(\tan\theta)^3} \\
&= \int \frac{\sec\theta}{\tan^2\theta} d\theta \\
&= \int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} d\theta \\
&= \int \frac{1}{\sin\theta} d\theta \\
&= \int \csc\theta d\theta \\
&= -\ln \mid \csc\theta + \cot\theta \mid + C
\end{aligned}$$

2. Determine if the following integrals converge or diverge. Give reasons for your answers. If the integral converges, find it's value if possible.

(a) $\displaystyle\int_0^1 \ln x dx$

$$= \lim_{t\to 0}\int_t^1 \ln x dx$$

$$u = \ln x \qquad\qquad dv = dx$$

$$du = \frac{1}{x}dx \qquad\qquad v = x$$

$$\begin{aligned}
&= \lim_{t\to 0}\left(x\ln x - \int x\cdot\frac{1}{x}dx\right) \\
&= \lim_{t\to 0}\left(x\ln x - x\right|_t^1 \\
&= \lim_{t\to 0} -1 - t\ln t + t \\
&= -1 - \lim_{t\to 0}\frac{\ln t}{\frac{1}{t}} \\
&=^{LH} -1 - \lim_{t\to 0}\frac{\frac{1}{t}}{-\frac{1}{t^2}} \\
&= -1 + \lim_{t\to 0}\frac{t^2}{t}
\end{aligned}$$

$$= -1 - \lim_{t\to 0} t$$
$$= 1$$

Therefore the integral converges.

(b) $\displaystyle\int_3^5 \frac{1}{x-4}dx$

$$= \int_3^4 \frac{1}{x-4}dx + \int_4^5 \frac{1}{x-4}dx$$

We will do the first integral first:

$$\int_3^4 \frac{1}{x-4}dx = \lim_{t\to 4^-} \int_3^t \frac{1}{x-4}dx$$
$$= \lim_{t\to 4^-} (\ln \mid x-4 \mid|_3^t$$
$$= \lim_{t\to 4^-} \ln \mid t-4 \mid - \ln \mid -1 \mid$$
$$= \lim_{t\to 4^-} \ln \mid t-4 \mid -0$$
$$= -\infty$$

Therefore, since one part of the integral diverges, the entire integral diverges.

(c) $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

$$= \lim_{t\to 3^-} \int_0^t \frac{dx}{\sqrt{9-x^2}}$$
$$= \lim_{t\to 3^-} \left(\frac{1}{3}\sin^{-1}\left(\frac{x}{3}\right)\right)_0^t$$
$$= \lim_{t\to 3^-} \frac{1}{3}\sin^{-1}\left(\frac{t}{3}\right) - \frac{1}{3}\sin^{-1} 0$$
$$= \frac{1}{3}\frac{\pi}{2} - 0$$
$$= \frac{\pi}{6}$$

so the integral converges.

(d) $\displaystyle\int_0^\infty \frac{2dx}{x^2-2x}$

$$= \int_0^1 \frac{2dx}{x^2-2x} + \int_1^2 \frac{2dx}{x^2-2x} + \int_2^3 \frac{2dx}{x^2-2x} + \int_3^\infty \frac{2dx}{x^2-2x}$$

We will look at the last one:

$$\int_3^\infty \frac{2dx}{x^2-2x} = \lim_{t\to\infty} 2\int_3^t \frac{dx}{x(x-2)}$$
$$= \lim_{t\to\infty} 2\int_3^t \frac{A}{x} + \frac{B}{x-2}dx$$

solving for the coefficients:

$$A(x-2)+Bx=1$$

If $x=0 \Rightarrow -2A=1 \Rightarrow A=-\frac{1}{2}$
If $x=2 \Rightarrow 2B=1 \Rightarrow B=\frac{1}{2}$

$$=\lim_{t\to\infty} 2\int_3^t \frac{-\frac{1}{2}}{x}+\frac{\frac{1}{2}}{x-2}dx$$

$$=\lim_{t\to\infty} 2\left(-\frac{1}{2}\ln\mid x\mid+\frac{1}{2}\ln\mid x-2\mid\right)_3^t$$

$$=\lim_{t\to\infty}\left(-\ln\mid x\mid+\ln\mid x-2\mid\right)_3^t$$

$$=\lim_{t\to\infty}\left(\ln\left|\frac{x-2}{x}\right|\right)_3^t$$

$$=\lim_{t\to\infty}\ln\left|\frac{t-2}{t}\right|-\ln\frac{1}{3}$$

$$=\ln 1-\ln\frac{1}{3}=-\ln\frac{1}{3}$$

Let us look at

$$\int_0^1 \frac{2dx}{x^2-2x}=\lim_{t\to 0^+}\int_t^1 \frac{2dx}{x^2-2x}$$

$$=\lim_{t\to 0^+}\int_t^1 \frac{-\frac{1}{2}}{x}+\frac{\frac{1}{2}}{x-2}dx$$

$$=\lim_{t\to 0^+}\left(\ln\left|\frac{x-2}{x}\right|\right)_t^1$$

$$=\lim_{t\to 0^+}\ln\left|\frac{-1}{1}\right|-\ln\left|\frac{t-2}{t}\right|$$

$$=0-\lim_{t\to 0^+}\ln\left|\frac{t}{t}-\frac{2}{t}\right|$$

$$=0-\infty=-\infty$$

So this integral diverges.

(e) $\displaystyle\int_1^\infty \frac{3x-1}{4x^3-x^2}$

$$=\lim_{t\to\infty}\int_1^t \frac{3x-1}{x^2(2x-1)}dx$$

$$=\lim_{t\to\infty}\int_1^t \frac{A}{x}+\frac{B}{x^2}+\frac{C}{2x-1}dx$$

solving for the coefficients:

$$Ax(2x-1)+B(2x-1)+Cx^2=3x-1$$

If $x=0 \Rightarrow -B=-1 \Rightarrow B=1$

If $x = \frac{1}{2} \Rightarrow \frac{1}{4}C = \frac{3}{2} - 1 = \frac{1}{2} \Rightarrow C = 2$

$$2Ax^2 - Ax + 2Bx - B + Cx^2 = 3x - 1$$
$$(2A + C)x^2 - (2B - A)x - B = 3x - 1$$
$$(2A + 2)x^2 + (A - 2)x - 1 = 3x - 1$$
$$A - 2 = 3 \Rightarrow A = 5$$

$$= \lim_{t\to\infty} \int_1^t \frac{5}{x} + \frac{1}{x^2} + \frac{2}{2x-1} dx$$
$$= \lim_{t\to\infty} \left(5\ln \mid x \mid - \frac{1}{x} + \ln \mid 2x - 1 \mid\right)_1^t$$
$$= \lim_{t\to\infty} 5\ln \mid t \mid - \frac{1}{t} + \ln \mid 2x - 1 \mid - 0 + 1 + 0$$
$$= \infty$$

so the integral diverges.

(f) $\int_0^\infty x^2 e^{-x} dx$

$$= \lim_{t\to\infty} \int_0^t x^2 e^{-x} dx$$
$$u = x^2 \qquad dv = e^{-x}dx$$
$$du = 2xdx \qquad v = -e^{-x}$$
$$= \lim_{t\to\infty} -x^2 e^{-x} + 2\int_0^t xe^{-x} dx$$
$$u = x \qquad dv = e^{-x}dx$$
$$du = dx \qquad v = -e^{-x}$$
$$= \lim_{t\to\infty} -x^2 e^{-x} + 2\left(-xe^{-x} + \int_0^t e^{-x} dx\right)$$
$$= \lim_{t\to\infty} \left(-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right)_0^t$$
$$= \lim_{t\to\infty} \left(-t^2 e^{-t} - 2te^{-t} - 2e^{-t}\right) - (-2) = 0 + 2 = 0$$

so the integral converges.

(g) $\int_1^\infty \frac{e^{-t}}{\sqrt{t}} dt$

$$0 < \frac{e^{-t}}{\sqrt{t}} \le e^{-t} \text{ for } t \ge 1$$
$$\int_1^\infty e^{-t} = \lim_{s\to\infty} -e^{-t} \mid_1^s$$
$$= \lim_{s\to\infty} -e^{-s} + \frac{1}{e}$$

$$= \frac{1}{e}$$

Since the larger integral converges, $\int_1^\infty \frac{e^{-t}}{\sqrt{t}} dt$ also converges, although we can't find it's value.

3. Calculate the following limits:

   (a) $\lim_{x\to 0} \frac{x \sin x}{1 - \cos x}$

$$=^{LH} \lim_{x\to 0} \frac{1 \sin x + x \cos x}{\sin x}$$

$$=^{LH} \lim_{x\to 0} \frac{-x \sin x + \cos x + \cos x}{\cos x}$$

$$= \lim_{x\to 0} \frac{2\cos x - x \sin x}{\cos x} = \frac{2-0}{1} = 2$$

   (b) $\lim_{x\to\infty} x^{\frac{1}{1-x}}$

$$f(x) = x^{\frac{1}{1-x}}$$

$$\ln f(x) = \frac{1}{1-x} \ln x = \frac{\ln x}{1-x}$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{1-x}$$

$$=^{LH} \lim_{x\to\infty} \frac{\frac{1}{x}}{-1}$$

$$= \lim_{x\to\infty} -\frac{1}{x} = 0$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^{\lim_{x\to\infty} \ln f(x)} = e^0 = 1$$

   (c) $\lim_{t\to 0} \frac{\tan 3t}{\tan 5t}$

$$=^{LH} \lim_{t\to 0} \frac{3 \sec^2 3t}{5 \sec^2 5t} = \frac{3}{5}$$

4. Solve the following differential equations

   (a) $\frac{dy}{dx} = -\frac{y \ln y}{1+x^2}$ where $y(0) = e^2$

$$\frac{1}{y \ln y} dy = -\frac{1}{1+x^2} dx$$

$$\ln(\ln y) = -\tan^{-1} x + C$$

$$y(0) = e^2$$

$$\ln(\ln(e^2)) = \tan^{-1}(0) + C$$

$$\ln 2 = C$$

$$\ln(\ln y) = -\tan^{-1} x + \ln 2$$

(b) $\dfrac{dy}{dx} + \left(\dfrac{2}{x+1}\right) y = \dfrac{x}{x+1}$

$$\mu(x) = e^{\int \frac{2}{x+1}} = e^{2\ln|x+1|} = e^{\ln|x+1|^2} = (x+1)^2$$

$$\begin{aligned} y &= \frac{1}{\mu(x)} \int \mu(x) g(x) dx \\ &= \frac{1}{(x+1)^2} \int (x+1)^2 \frac{x}{x+1} dx \\ &= \frac{1}{(x+1)^2} \int x(x+1) dx \\ &= \frac{1}{(x+1)^2} \int x^2 + x dx \\ &= \frac{1}{(x+1)^2} \left(\frac{x^3}{3} + \frac{x^2}{2} + C \right) \end{aligned}$$

(c) $x\dfrac{dy}{dx} + 2y = x^2 + 1$

$$\begin{aligned} \frac{dy}{dx} + \frac{2}{x} y &= x + \frac{1}{x} \\ \mu(x) &= e^{\int \frac{2}{x} dx} = e^{\ln x^2} = x^2 \\ y &= \frac{1}{x^2} \int x^2 \left(x + \frac{1}{x} \right) dx \\ &= \frac{1}{x^2} \int x^3 + x dx \\ &= \frac{1}{x^2} \left(\frac{x^4}{4} + \frac{x^2}{2} + C \right) \end{aligned}$$

(d) $xy' + y = x \cos x$

$$\begin{aligned} y' + \frac{1}{x} y &= \cos x \\ \mu(x) &= e^{\int \frac{1}{x} dx} = e^{\ln x} = x \\ y &= \frac{1}{x} \int x \cos x dx \end{aligned}$$

To do the integral, let

$$\begin{aligned} u &= x & dv &= \cos x dx \\ du &= dx & v &= -\sin x \end{aligned}$$

$$y = \frac{1}{x} \left(-x \sin x + \int \sin x dx \right)$$

$$y = \frac{1}{x} \left(-x \sin x - \cos x + C \right)$$

5. Find the lengths of the following curves on the given intervals;

(a) $x = y^{\frac{2}{3}}, 1 \leq y \leq 8$

$$\frac{dx}{dy} = \frac{2}{3}y^{-\frac{1}{3}}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{4}{9}y^{-\frac{2}{3}}$$

$$L = \int_1^8 \sqrt{1 + \frac{4}{9y^{\frac{2}{3}}}}dy$$

$$= \int_1^8 \sqrt{\frac{9y^{\frac{2}{3}} + 4}{9y^{\frac{2}{3}}}}dy$$

$$= \int_1^8 \frac{\sqrt{9y^{\frac{2}{3}}}}{3y^{\frac{1}{3}}}dy$$

$$\frac{1}{3}\int_1^8 \sqrt{9y^{\frac{2}{3}} + 4}\left(y^{-\frac{1}{3}}\right) dy$$

$$u = 9y^{\frac{2}{3}} \qquad du = 6y^{-\frac{1}{3}}$$

$$= \frac{1}{18}\int_{13}^{40} u^{\frac{1}{2}}du$$

$$= \frac{1}{18}\left(\frac{2}{3}u^{\frac{3}{2}}\right)_{13}^{40}$$

$$= \frac{1}{27}\left(40^{\frac{3}{2}} - 13^{\frac{3}{2}}\right) \approx 7.634$$

(b) $x = 5\cos t - \cos 5t, y = 5\sin t - \sin 5t, 0 \leq t \leq \frac{\pi}{2}$

$$\frac{dx}{dt} = -5\sin t + 5\sin 5t \qquad \frac{dy}{dt} = 5\cos t - 5\cos 5t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(5\sin 5t - 5\sin t)^2 + (5\cos t - t\cos 5t)^2}$$

$$= 5\sqrt{\sin^2 5t - 2\sin t\sin 5t + \sin^2 t + \cos^2 t - 2\cos t\cos 5t + \cos^2 5t}$$

$$= 5\sqrt{2 - 2\left(\sin t\sin 5t + \cos t\cos 5t\right)}$$

$$= 5\sqrt{2\left(1 - \cos 4t\right)}$$

$$= 5\sqrt{4\left(\frac{1}{2}\right)\left(1 - \cos 4t\right)}$$

$$= 10\sqrt{\sin^2 2t}$$

$$= 10 \mid \sin 2t \mid = 10\sin 2t \text{ since } 0 \leq t \leq \frac{\pi}{2}$$

$$L = \int_0^{\frac{\pi}{2}} 10\sin 2t dt = (-5\cos 2t)_0^{\frac{\pi}{2}} = (-5)(-1) - (-5)(1) = 10$$