

Math 244
Fall, 2004

The following problems are the homework problems for the course. Problems marked with a **(C)** are computational problems and are not eligible for presentation during class. However, the computational problems may appear on quizzes and exams, and are assigned to assure that you have mastered the basic skills required in each section, in addition to having mastered the theory of the topics discussed.

Remember that you are not to consult with other students except on computational problems or problems already presented in class. You can use the textbook for reference, but all solutions to assigned problems should be in your own words, not parroted from the book or other sources.

You can use previous problems on later problems, if they are helpful. However, you cannot use a later problem for help. Part of the challenge of these problems is to find and rectify any typographical errors.

1. (Katie/Amanda) Find the unit vectors that are parallel to the vector $2\vec{i} - 3\vec{j}$.
2. (Courtney) Find a vector of length 2 whose direction is the opposite of the direction of the vector $-\vec{i} + 2\vec{j}$.
3. (a) (Sarah) Find the unit vectors that are tangent and normal to the curve $x^2 + 2y^2 = 6$ at the point $(2, 1)$.
(b) (Amanda) If $\vec{v} = \langle v_1, v_2 \rangle$ is a tangent vector, is the vector $\langle -v_2, v_1 \rangle$ always normal?
4. (Ashley) Find the unit vectors that are tangent and normal to the curve $x^2 - 6xy + 8y^2 - 2x - 1 = 0$ at the point $(1, 1)$.
5. (Eunice) Find the unit vectors that are tangent and normal to the curve $\int_e^x \ln(\ln t) dt$ at the point $(e, 0)$.
6. (Clint) An airplane is flying in the direction 25° west of north at 800 km per hour. Find the component form of the velocity of the airplane, assuming that the positive x -axis represents due east and the positive y -axis represents due north.
7. (Cherie) A bird flies from its nest 5 km in the direction of 60° north of east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast and lands atop a telephone pole. Place an xy -coordinate system so that the origin lands on the bird's nest, the x -axis points east and the y -axis points north.
 - (a) At what point is the tree located?
 - (b) At what point is the telephone pole located?
8. **(C)** For the vectors $\vec{v} = 2\vec{i} + 10\vec{j}$ and $\vec{u} = 2\vec{i} + 2\vec{j}$, find:
 - (a) $\vec{v} \cdot \vec{u}$

- (b) $|\vec{v}|$
 (c) $|\vec{u}|$
 (d) the cosine of the angle between \vec{v} and \vec{u}
 (e) the scalar component of \vec{u} in the direction \vec{v}
 (f) the vector projection, $proj_{\vec{v}}\vec{u}$
9. (C) For the vectors $\vec{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ and $\vec{u} = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$, find:
- (a) $\vec{v} \cdot \vec{u}$
 (b) $|\vec{v}|$
 (c) $|\vec{u}|$
 (d) the cosine of the angle between \vec{v} and \vec{u}
 (e) the scalar component of \vec{u} in the direction \vec{v}
 (f) the vector projection, $proj_{\vec{v}}\vec{u}$
10. (C) Find the measures of the angles between the diagonals of the rectangle whose vertices are $A = (1, 0)$, $B = (0, 3)$, $C = (3, 4)$ and $D = (4, 1)$.
11. (Quiz problem) Suppose that AB is the diameter of a circle with center O and that C is a point on one of the two arcs joining A and B . Show that \overrightarrow{CA} and \overrightarrow{CB} are orthogonal.
12. (Leslie and Patrick) Show that the diagonals of a rhombus (parallelogram with sides of equal length) are perpendicular.
13. (Sarah) Show that squares are the only rectangles with perpendicular diagonals.
14. (Amanda and Ashley) Prove that a parallelogram is a rectangle if and only if its diagonals are equal in length.
15. (Leslie) (Cauchy-Schwartz Inequality) Use the fact that $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$ to show that the inequality $|\vec{u} \cdot \vec{v}| \leq |\vec{u}| |\vec{v}|$ holds for all vectors \vec{u} and \vec{v} .
16. (Courtney) Show that $\vec{v} = a\vec{i} + b\vec{j}$ is perpendicular to the line $ax + by = c$.
17. (C) Let $\vec{r}(t) = (2 \ln(t+1))\vec{i} + (t^2)\vec{j}$ be the position vector of a particle in the plane at time t .
- (a) Sketch a graph of the particle
 (b) Find the velocity vector.
 (c) Find the acceleration vector.
 (d) Find the particle's speed and direction of motion at $t = 1$.
 (e) Write the particle's velocity at $t = 1$ as the product of its speed and direction.
18. (Robert) Let the position vector of a particle in the plane at time t be given by $r(t) = (\sin t)\vec{i} + t\vec{j}$ for $t \geq 0$. Find the time(s) when the velocity and acceleration vectors are perpendicular.
19. (Troy) Let the position vector of a particle in the plane at time t be given by $r(t) = (3t + 1)\vec{i} + (t^2)\vec{j}$. Find the angle between the velocity and acceleration vectors at $t = 0$.

20. (Amanda) A particle moves in the plane so that its velocity and position vectors are always orthogonal. Show that the particle moves in a circle centered at the origin.
21. (Amanda) A particle moves around the unit circle in the xy -plane. Its position at time t is $\vec{r} = x\vec{i} + y\vec{j}$, where x and y are differentiable functions of t . Find $\frac{dy}{dt}$ if $\vec{v} \cdot \vec{r} = y$. Is the motion clockwise or counterclockwise?
22. (C) Find $\lim_{t \rightarrow 3} \left(t\vec{i} + \frac{t^2 - 9}{t^2 + 3t}\vec{j} \right)$. Is the function continuous at $t = 3$?
23. (C) Find the equation of the tangent and normal lines to the curve

$$\vec{r}(t) = (2 \cos t - 3)\vec{i} + (3 \sin t + 1)\vec{j} \text{ at } t = \frac{\pi}{4}$$

24. (C) Find $\int_1^2 \left((6 - 6t\vec{i} + 3\sqrt{t}\vec{j}) dt \right)$.

25. (C) Find $\int \left(\frac{1}{t}\vec{i} + \frac{1}{5-t}\vec{j} \right) dt$

26. (C) Solve the initial value problem for \vec{r} as a vector function of t given that

$$\frac{d^2\vec{r}}{dt^2} = -\vec{i} - \vec{j} \quad \vec{r}(0) = 10\vec{i} + 10\vec{j} \quad \left. \frac{d\vec{r}}{dt} \right|_{t=0} = \vec{0}$$

27. (The class) The position of a particle in the plane at time t is given by $\vec{r}(t) = (1 - \cos t)\vec{i} - (t - \sin t)\vec{j}$. Find the distance the particle travels along the path from $t = 0$ to $t = \frac{2\pi}{3}$.
28. (Riley) At time $t = 0$ a particle is located at the point $(1, 2)$. It travels in a straight line to the point $(4, 1)$, has speed 2 at $(1, 2)$ and constant acceleration $3\vec{i} - \vec{j}$. Find an equation for the position vector $\vec{r}(t)$ of the particle at time t .
29. (Patrick, except a) The position of a kite is given by $\vec{r}(t) = \frac{t}{8}\vec{i} - \frac{3}{64}t(t - 160)\vec{j}$ where $t \geq 0$ is measured in seconds and distance is measured in meters.
- (a) (Courtney) How long is the kite above ground?
- (b) How high is the kite at $t = 40$ seconds?
- (c) At what rate is the kite's altitude increasing at $t = 40$ seconds?
- (d) At what time does the kite start to lose altitude?

30. (Amanda) Let \vec{v} be a differentiable vector function of t . Show that if $\vec{v} \cdot \left(\frac{d\vec{v}}{dt} \right) = \vec{0}$ for all t , then $|\vec{v}|$ is constant.

31. (Emily) Prove that if \vec{u} is the vector function with constant value \vec{C} then $\frac{d\vec{u}}{dt} = \vec{0}$.

32. (Leslie) Show that if $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$ is differentiable at $t = c$ then \vec{r} is continuous at c .

33. (Cherie) A projectile is fired at a speed of 840 m/sec at an angle of 60° . How long will it take to get 21 km downrange.

34. (Robert) A baseball is thrown from the stand 32 feet above the field at an angle of 30° up from the horizontal. When and how far away will the ball strike the ground if its initial speed is 32 ft/sec?
35. (Clint) A human cannonball is to be fired with an initial speed of $v_0 = \frac{80\sqrt{10}}{2}$ feet per second. The circus performer hopes to land on a special cushion located 200 feet downrange at the same height as the muzzle of the cannon. The circus is being held in a large room with a flat ceiling 75 feet higher than the muzzle. Can the performer be fired to the cushion without striking the ceiling? if so, what should the cannon's angle of elevation be?
36. (Riley and Amanda) A baseball hit by a Boston Red Sox player at a 20° angle from 3 feet above the ground just cleared the left end of the "Green Monster," the left-field wall in Fenway Park. This wall is 37 feet high and 315 feet from home plate.
- (a) What was the initial speed of the ball?
- (b) How long did it take the ball to reach the wall?
37. (Ashley) In Moscow in 1987, Natalya Lisouskaya set a women's world record by putting an 8 pound 13 ounce shot 73 feet 10 inches. Assuming that she launched the shot at a 40° angle to the horizontal from 6.5 feet above the ground, what was the shot's initial speed?
38. (C) Be able to do problems 1-48 in section 9.5
39. (Leslie) Explain why every vertical line in the plane has a polar equation of the form $r = a \sec \theta$
40. (August) Find an analogous polar equation for horizontal lines.
41. (Courtney) Find a formula for the distance between points (r_1, θ_1) and (r_2, θ_2) . Be able to explain why your formula works.
42. Find the maximum height above the x -axis of the cardioid $r = 2(1 + \cos \theta)$.
43. (C) Find the slope of the curve $r = 2 - 3 \sin \theta$ at $(2, \pi)$.
44. (C) Find the tangent lines to $r = 2 \sin 2\theta$ at the pole for $0 \leq \theta \leq 2\pi$.
45. (C) Find equations for the horizontal and vertical tangents to the curve $r = 3 - 4 \cos \theta$ for $0 \leq \theta \leq 2\pi$.
46. (C) Find the area of the region inside the cardioid $r = a(1 + \cos \theta)$ for $a > 0$.
47. (C) Find the area inside the six-leaved rose $r^2 = 2 \sin 3\theta$.
48. (C) Find the area shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos \theta)$.
49. (C) Find the area inside the circle $r = 6$ above the line $r = 3 \csc \theta$.
50. (C) Find the length of the spiral $r = \frac{e^\theta}{\sqrt{2}}$ for $0 \leq \theta \leq \pi$.
51. (Joel) Recall the definition of average value of a function from Calc. II. Find the average value of $r = a(1 - \cos \theta)$ for $a > 0$.

- 51.5. (Amanda) Recall that for parametric equations, arc length is $L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
for $\alpha \leq t \leq \beta$. Use this to show that the length of the curve in a polar graph is $L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$
52. (Ashley) What can be said about the relative lengths of the curves $r = f(\theta)$ and $r = 2f(\theta)$ where $\alpha \leq \theta \leq \beta$?
53. (C) Be able to do problems 1-40 in Section 10.1.
54. (C) Find a vector of magnitude 7 in the direction of $\vec{v} = 12\vec{i} - 5\vec{k}$.
55. (C) Consider the points $P_1(3, 4, 5)$ and $P_2(2, 3, 4)$.
- Find the distance between P_1 and P_2 .
 - Find the direction of $\overrightarrow{P_1P_2}$
 - Find the midpoint of the line segment P_1P_2 .
56. (C) Find the center and the radius of the sphere given by $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9$.
57. (Emily) Let $ABCD$ be a general, not necessarily planar, quadrilateral in space. Show that the two segments joining the midpoints of the opposite sides of $ABCD$ bisect each other.
58. (Riley) Suppose that A , B , and C are vertices of a triangle and that a , b and c , respectively, are the midpoints of the opposite sides. Show that $\overrightarrow{Aa} + \overrightarrow{Bb} + \overrightarrow{Cc} = 0$.
59. (C) For the vectors $\vec{v} = \vec{i} + \vec{j}$ and $\vec{u} = \sqrt{2}\vec{i} + \sqrt{3}\vec{j} + 2\vec{k}$,
- Find $\vec{v} \cdot \vec{u}$
 - $|\vec{v}|$
 - $|\vec{u}|$
 - the cosine of the angle between \vec{v} and \vec{u}
 - the scalar component of \vec{u} in the direction \vec{v}
 - the vector projection, $proj_{\vec{v}}\vec{u}$
60. (C) Write $\vec{u} = 8\vec{i} + 4\vec{j} - 12\vec{k}$ as the sum of a vector parallel to $\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$ and a vector orthogonal to \vec{v} .
61. (C) Find the length and direction of $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ for $\vec{u} = -8\vec{i} - 2\vec{j} - 4\vec{k}$ and $\vec{v} = 2\vec{i} + 2\vec{j} + \vec{k}$.
62. (C) Find the area of the triangle determined by $P(-2, 2, 0)$, $Q(0, 1, -1)$ and $R(-1, 2, -2)$.
63. (C) Find the volume of the parallelepiped determined by $\vec{u} = \vec{i} - \vec{j} + \vec{k}$, $\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{w} = -\vec{i} + 2\vec{j} - \vec{k}$.
64. (Eunice) Let $\vec{u} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{v} = -\vec{i} + \vec{j} + \vec{k}$, $\vec{w} = \vec{i} + \vec{k}$, and $\vec{r} = -\frac{\pi}{2}\vec{i} - \pi\vec{j} + \frac{\pi}{2}\vec{k}$. Which vectors are perpendicular? parallel?

65. (August a-d, Sarah f-h, Katie i-l, Leslie m-p) Which of the following are always true and which are not always true?

- (a) $|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$
- (b) $\vec{u} \cdot \vec{u} = |\vec{u}|$
- (c) $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$
- (d) $\vec{u} \times (-\vec{u}) = \vec{0}$
- (e) $\vec{u} \times \vec{v} = \vec{v} \times \vec{u}$
- (f) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- (g) $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
- (h) $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$
- (i) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- (j) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- (k) $(-u) \times \vec{v} = -(\vec{u} \times \vec{v})$
- (l) $(c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) = c(\vec{u} \cdot \vec{v})$
- (m) $(c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$
- (n) $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
- (o) $(\vec{u} \times \vec{u}) \cdot \vec{u} = 0$
- (p) $\vec{u} \times \vec{v} \cdot \vec{u} = \vec{v} \cdot (\vec{u} \times \vec{v})$

66. (Clint) Let \vec{u} , \vec{v} , and \vec{w} be vectors. Which of the following make sense, and which do not?

- (a) $(\vec{u} \times \vec{v}) \cdot \vec{w}$
- (b) $\vec{u} \times (\vec{v} \cdot \vec{w})$
- (c) $\vec{u} \times (\vec{v} \times \vec{w})$
- (d) $\vec{u} \cdot (\vec{v} \cdot \vec{w})$

Exam 1 covers to this point

Exam 2 begins here

67. (C) Find the vector and parametric equations for the line through $P(-2, 0, 3)$ and $Q(3, 5, -2)$.
68. (C) Find the vector and parametric equations for the line through $(1, 1, 1)$ parallel to the z -axis.
69. (C) Find the line through $(2, 3, 0)$ perpendicular to the vectors $\vec{u} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{v} = 3\vec{i} + 4\vec{j} + 5\vec{k}$.
70. (C) Find the equation of the plane through $(0, 2, -1)$ normal to $\vec{n} = 3\vec{i} - 2\vec{j} - \vec{k}$.
71. (C) Find the equation of the plane through $(1, -1, 3)$ parallel to the plane $3x + y + z = 7$.
72. (C) Find the equation of the plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$.
73. (C) Find the distance from the line $x = 2 + t$, $y = 1 + t$, $z = -\frac{1}{2} - \frac{1}{2}t$ to the plane $x + 2y + 6z = 10$.
74. If \vec{n}_1 and \vec{n}_2 are the normals to two planes, show that the angles between the planes is

$$\theta = \cos^{-1} \left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} \right)$$

75. Find equations for the line in the plane $z = 3$ that makes an angle of $\frac{\pi}{6}$ radians with \vec{i} and an angle of $\frac{\pi}{3}$ radians with \vec{j} .
76. How can you tell when two planes $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$ are parallel? perpendicular?
77. Suppose that L_1 and L_2 are disjoint nonparallel lines. Is it possible for a nonzero vector to be perpendicular to both L_1 and L_2 ?
78. (C) Be able to do problems 1-12 in section 10.4.
79. (C) Be able to do problems 1-24 in section 10.5.
80. Find the parametric equations for the line that is tangent to the curve $\vec{r}(t) = (a \sin t)\vec{i} + (a \cos t)\vec{j} + bt\vec{k}$ at $t = 2\pi$.
81. At time $t = 0$ a particle is located at the point $(1, 2, 3)$. It travels in a straight line to the point $(4, 1, 4)$, has speed 2 at $(1, 2, 3)$ and constant acceleration $3\vec{i} - \vec{j} + \vec{k}$. Find an equation for the position vector $\vec{r}(t)$ of the particle at time t .
82. Show that the vector-valued function

$$\vec{r}(t) = (2\vec{i} + 2\vec{j} + \vec{k}) + (\cos t) \left(\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j} \right) + (\sin t) \left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k} \right)$$

describes the motion of a particle moving on the circle of radius 1 centered at the point $(2, 2, 1)$ and lying in the plane $x + y - 2z = 2$.

83. Let \vec{v} be a differentiable vector function of t . Show that if $\vec{v} \cdot \left(\frac{d\vec{v}}{dt} \right) = 0$ for all t , then $|\vec{v}|$ is constant.

84. Show that the vector function \vec{r} defined by the rule $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ is continuous at $t = t_0$ if and only if f, g and h are continuous at t_0 .
85. (C) Find the length of the curve $\vec{r}(t) = (2 \cos t)\vec{i} + (2 \sin t)\vec{j} + \sqrt{5}t\vec{k}$ for $0 \leq t \leq \pi$.
86. Find the unit tangent vector to the curve $\vec{r}(t) = (2 \cos t)\vec{i} + (2 \sin t)\vec{j} + \sqrt{5}t\vec{k}$ at any time t .
87. (C) Find the length of the curve $\vec{r}(t) = (t \cos t)\vec{i} + (t \sin t)\vec{j} + \left(\frac{2\sqrt{2}}{3}\right)t^{\frac{3}{2}}\vec{k}$ for $0 \leq t \leq \pi$.
88. Find the unit tangent vector to the curve $\vec{r}(t) = (t \cos t)\vec{i} + (t \sin t)\vec{j} + \left(\frac{2\sqrt{2}}{3}\right)t^{\frac{3}{2}}\vec{k}$ at any time t .
89. Find the point on the curve $\vec{r}(t) = (5 \sin t)\vec{i} + (5 \cos t)\vec{j} + 12t\vec{k}$ at a distance of 26π units along the curve from the point $(0, 5, 0)$ when $t = 0$ in the direction of increasing arc length.
90. (C) Find T , N , and κ for the plane curve $\vec{r}(t) = t\vec{i} + (\ln \cos t)\vec{j}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.
91. (C) Find the length of the curve $\vec{r}(t) = (\sqrt{2}t)\vec{i} + (\sqrt{2}t)\vec{j} + (1-t^2)\vec{k}$ from $(0, 0, 1)$ to $(\sqrt{2}, \sqrt{2}, 0)$.
92. Find an equation for the circle of curvature of the curve $\vec{r}(t) = t\vec{i} + \sin t\vec{j}$ at the point $(\frac{\pi}{2}, 1)$.
93. Explain what T , N , and B measure. Why are they important?
94. Explain what κ and τ measure. Why are they important?
95. (C) Find T , N , B , κ , and τ for the space curve
- $$\vec{r}(t) = (3 \sin t)\vec{i} + (3 \cos t)\vec{j} + 4t\vec{k}$$
96. (C) Find T , N , B , κ , and τ for the space curve
- $$\vec{r}(t) = (\cos t + t \sin t)\vec{i} + (\sin t - t \cos t)\vec{j} + 3\vec{k}$$
97. (C) Find T , N , B , κ , and τ for the space curve
- $$\vec{r}(t) = (\cos^3 t)\vec{i} + (\sin^3 t)\vec{j}$$
98. The speedometer on your car reads a steady 35 mph. Could you be accelerating? Explain.
99. Can anything be said about the speed of a particle whose acceleration is always orthogonal to its velocity? Explain.
100. Show that a moving particle will move in a straight line if the normal component of its acceleration is zero.
101. Show that κ and τ are both zero for the line
- $$\vec{r}(t) = (x_0 + At)\vec{i} + (y_0 + Bt)\vec{j} + (z_0 + Ct)\vec{k}$$
102. Show that the curvature of a smooth curve $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$ defined by twice-differentiable functions $x = f(t)$ and $y = g(t)$ is given by the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$$