Math 244
Fall, 2003 Name: **Solutions**
Review of Multiple Integrals

Conversion Chart

Cartesian	**Polar**	**Cylindrical**	**Spherical**
$dA = dx\ dy$	$dA = r\ dr\ d\theta$		
$dV = dx\ dy\ dz$		$dV = dz\ r\ dr\ d\theta$	$dV = \rho^2 \sin\phi d\rho\ d\phi\ d\theta$
	$x = r\cos\theta$	$x = r\cos\theta$	$r = \rho\sin\phi$
	$y = r\sin\theta$	$y = r\sin\theta$	$z = \rho\cos\phi$
	$x^2 + y^2 = r^2$	$z = z$	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$		$x = \rho\sin\phi\cos\theta$
			$y = \rho\sin\phi\sin\theta$
			$z = \rho\cos\phi$
			$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$

1. Find the volume of the solid enclosed by the cardioid of revolution $\rho = 1 - \cos\phi$

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos\phi} \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} (1 - \cos\phi)^3 \sin\phi\ d\phi\ d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \left(\frac{(1-\cos\phi)^4}{4} \right|_0^{\pi} d\theta \\ &= \frac{1}{12}(2^4) \int_0^{2\pi} d\theta \\ &= \frac{4}{3}(2\pi) \\ &= \frac{8\pi}{3} \end{aligned}$$

2. Find the volume of the solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2+y^2}$ (For a picture, see §12.6, number 37)

$$\begin{aligned}
V &= \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos\phi} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta \\
&= \frac{8}{3} \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3\phi \sin\phi d\phi d\theta \\
&= \frac{8}{3} \int_0^{2\pi} \left(-\frac{\cos^4\phi}{4} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right. d\theta \\
&= \left(\frac{8}{3}\right)\left(\frac{1}{16}\right) \int_0^{2\pi} d\theta \\
&= \frac{1}{6}(2\pi) \\
&= \frac{\pi}{3}
\end{aligned}$$

3. Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

$$\begin{aligned}
V &= 8 \int_0^{\frac{\pi}{2}} \int_1^{\sqrt{2}} \int_0^{\sqrt{2-r^2}} dz \; r \; dr \; d\theta \\
&= 8 \int_0^{\frac{\pi}{2}} \int_1^{\sqrt{2}} r\sqrt{2-r^2} \; dr \; d\theta \\
&= 8 \int_0^{\frac{\pi}{2}} \left(-\frac{1}{3}(2-r^2)^{\frac{3}{2}} \Big|_1^{\sqrt{2}} \right. d\theta \\
&= \frac{8}{3} \int_0^{\frac{\pi}{2}} d\theta \\
&= \frac{4\pi}{3}
\end{aligned}$$

OR

$$V = 2 \int_0^{2\pi} \int_1^{\sqrt{2}} \int_0^{\sqrt{2-r^2}} dz \; r \; dr \; d\theta$$

OR

$$V = \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} dz \; r \; dr \; d\theta$$

4. Find the volume of the region bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the $xy-$plane, and lying outside the cylinder $x^2 + y^2 = 1$.

The paraboloid intersects the $xy-$plane when

$$9 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 9$$

$$\begin{aligned} V &= 4\int_0^{\frac{\pi}{2}}\int_1^3\int_0^{9-r^2} dz\ r\ dr\ d\theta \\ &= 4\int_0^{\frac{\pi}{2}}\int_1^3 \left(9r - r^3\right) dr\ d\theta \\ &= 4\int_0^{\frac{\pi}{2}} \left(\frac{9r^2}{2} - \frac{r^4}{4}\right|_1^3 d\theta \\ &= 4\int_0^{\frac{\pi}{2}} \left(\frac{81}{4} - \frac{17}{4}\right) d\theta \\ &= 64\int_0^{\frac{\pi}{2}} d\theta \\ &= 32\pi \end{aligned}$$

OR

$$V = \int_0^{2\pi}\int_1^3\int_0^{9-r^2} dz\ r\ dr\ d\theta$$

5. Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 4$.

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^2\int_0^{4-r\sin\theta} dz\ r\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^2 \left(4r - r^2\sin\theta\right) dr\ d\theta \\ &= 8\int_0^{2\pi}\left(1 - \frac{\sin\theta}{3}\right) d\theta \\ &= 16\pi \end{aligned}$$

6. find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$ and below by the $xy-$plane.

$$\begin{aligned}
V &= 4\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2} dz\ dy\ dx \\
&= 4\int_0^2 \int_0^{\sqrt{4-x^2}} \left(4 - x^2\right)\ dy\ dx \\
&= 4\int_0^2 \left(4 - x^2\right)^{\frac{3}{2}}\ dx \\
&= \left(x(4-x^2)^{\frac{3}{2}} + 6x\sqrt{4-x^2} + 24\sin^{-1}\frac{x}{2}\right|_0^2 \\
&= 24\sin^{-1} 1 \\
&= 12\pi
\end{aligned}$$

OR

$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2} dz\ dy\ dx$$

OR

$$V = \int_0^{2\pi} \int_2^{2\sec\theta} \int_0^{4-r^2\cos^2\theta} dz\ r\ dr\ d\theta$$

(we think)