A Circular History of Knot Theory

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In the nineteenth century physicists were speculating about the underlying principles of atoms. In 1867, Lord Kelvin put forward a comprehensive theory of atoms which, through heuristic reasoning, seemed to explain several of the essential qualities of the chemical elements. Kelvin's theory conjectured that atoms were knotted tubes of ether. (To a topologist a knot in 3-space is any closed loop having no self-intersections and a link is any collection of non-intersecting closed loops.) The topological stability and the variety of knots were thought to mirror the stability of matter and the variety of chemical elements.

Kelvin's theory of vortex atoms was taken seriously for about two decades. Maxwell thought that ``it satisfies more of the conditions than any atom hitherto considers''. This theory inspired the celebrated Scottish physicist Peter Tait to undertake an extensive study and tabulation of knots in an attempt to understand when two knots were ``different''. (The later stages of this study were in collaboration with C. N. Little.) Tait's intuitive understanding of ``different'' and ``same'' is still a useful notion. Two knots are isotopic if one can be continuously manipulated in 3-space (no self-intersections allowed) until it looks like the other. The accompanying diagram shows a portion of Tait's study---an enumeration of knots and links in terms of the crossing number of a plane projection. If Kelvin's theory had been the correct foundation for the classification of the chemical elements, then Tait's knot table would have been the basis for a periodic table of elements. But Kelvin's theory was fundamentally mistaken and physicists lost interest in the Tait's work.

What the physicists abandoned, intrigued mathematicians, then and now, and the basic question is still the same: how do we tell when two knots are isotopically the same? (Research tip: Sometimes the most interesting problems can be found in someone else's trash.) This failed atomic theory also left in its wake the riches of Tait's tabulation---163 knot projections---and a rudimentary understanding of isotopic sameness in terms of how one projection could be continuously manipulated to look like another. This understanding of projection manipulation was summarized in a set of conjectures for knot projections, the famous Tait Conjectures.

To attack the Tait Conjectures and the basic question of sameness of knots, topologists developed knot invariants. An early example of a successful knot invariant is the Alexander polynomial, discovered by J. W. Alexander in 1927. The Alexander polynomial for the knot labeled 3_1 (the trefoil) is -(txt)+t -1 and the polynomial for 4_1 (the figure-eight) is -(txt)+3t-1. Since these two polynomials are different we know their associated knots are different. The Alexander polynomial was remarkable for how successful it was in distinguishing the knots in Tait's orginal table and it gave witness to how thorough a researcher Tait was. (Historical note: The last of the few duplications in the Tait/Little table was found in 1974 by Kenneth Perko, a New York lawyer and part-time topologist, while he was manipulating loops of rope on his living room floor. If a lawyer can do research in knot theory, it can't be that hard.) Unforunately, there are many knots with equivalent Alexander polynomial that can be shown to be isotopically different through the uses of other invariants.

So the search was on for more sensitive knot invariants that would detect when two knots were different. This led to alternate understandings of the notion of sameness. In particular, to a topologist there is no difference between the loops representing 4_1 and 5_1. What is different is the space away from these loops, that is the complement of the knot. Two topological spaces are homeomorphic if there is a bijective invertible continuous function that maps one space to the other. Thus, we have an alternate notion of sameness: if two knots/links have homeomorphic knot/link complements then they are homeomorphic knots/links. Now, it would seem that homeomorphic sameness would be weaker than isotopic sameness. And in fact, for link complements it is---there exist examples of links that are not isotopic, but have homeomorphic complements. But for knots a seminal result of Cameron Gordon and John Luecke showed that two knot are homeomorphic if and only if they are isotopic. In the vernacular of the knot theorist, a knot determines its complement.

Understanding that the principle object of study is the knot complement places knot theory inside the larger study of 3-manifolds. A 3-manifold is a space which locally (assume you are near sighted) looks like standard xyz-space and knot complements are readily seen as examples of 3-manifolds. It was through the study of 3-manifolds that in the 1970's knot theory began returning to its ancestoral roots in physics. To understand this we have to flashback to the 1860's work of Bernhard Riemann. Riemann was interested in relating geometric structures to the forces in physics. Building on Gauss' work, Riemann investigated three different geometric structures for 3-dimensional spaces---elliptic, euclidean, and hyperbolic. (Einstein's Theory of Relativity was built on Riemannian geometry.) Each of these distinct structures can be characterized by the behavior of triangles in planes. In elliptic 3-space, the interior angles of a triangle in a plane have a sum greater than 180 degrees. In Euclidean 3-space, the sum is 180 degrees and in hyperbolic 3-space the sum is less than 180 degrees. In 1978, William Thurston established sufficient conditions for when a 3-manifold possesses a hyperbolic structure. Surprisingly, except for a well understood subclass of knots, all knot complements possess a complete hyperbolic structure. (The beauty of Thurston's work is captured in the video Not Knot that is distributed by the American Mathematical Society and has been frequently viewed at Grateful Dead concerts.)

Thurston's work on hyperbolic structures firmly re-established knot theory's connections with physics. In the 1980's, through some totally unexpected routes, knot theory made further connections with its ancestral roots. In 1987 Vaughan Jones discovered a totally different polynomial invariant from that of Alexander using the theory of operator algebras. Within a short period of time, more than five new polynomial invariants generalizing the Jones polynomial were discovered. (One of these polynomial was simultaneously discovered by six different mathematicians and its name is an acronym of their last names---HOMFLY.) Moreover, Jones' polynomial quickly led to the proofs that established all of Tait's original conjectures on knot projections.

With this poliferation of new polynomials it was natural to ask whether any of these invariants had a natural extensions to all 3-manifolds. Two facts worked in favor of having such extensions: 1) all 3-manifolds can be describe in terms of knots and links via an operation called Dehn surgery; 2) there exists a set of moves, the Kirby calculus, that allow one to move between differing Dehn surgery descriptions of the same homeomorphic 3-manifold. Using the Kirby calculus as a means to generalizing the polynomial invariants, Edward Witten, a theoretical physicist, proposed new invariants for 3-manifolds. His invariants came out of the theoretical area of physics know as quantum field theory. These new invariants can be realized as certain averages of link polynomials obtained from a given Dehn surgery representation of the manifold.

Starting with the flawed theory of Kelvin's knotted vortex to the work of Thurston, Jones and Witten, knot theory has circled back to its ancestral orgins of theoretical physics.