Here are some good review questions. The actual exam might be of a different format, but these will help you understand the concepts covered on the exam. Try to do as many of these as you can without looking in your notes or book for guidance.

1. Calculate the following integrals:

   (a) $\int \csc u du$
   
   (b) $\displaystyle\int \frac{x-2}{x^2-4x}dx$
   
   (c) $\int x \sin x dx$
   
   (d) $\int x \cos x dx$
   
   (e) $\int e^{2x} \sin x dx$
   
   (f) $\displaystyle\int_1^9 \sqrt{x} \ln x dx$
   
   (g) $\displaystyle\int \frac{4x^2-7}{2x+3}dx$
   
   (h) $\int \cos^3 \theta d\theta$
   
   (i) $\int (\csc x - \tan x)^2 dx$
   
   (j) $\int \sqrt{1-\cos^2\theta} d\theta$
   
   (k) $\int (\csc x - \sec x)(\sin x + \cos x)dx$
   
   (l) $\int 10^{2\theta} d\theta$
   
   (m) $\int 2x \sin^{-1}(x^2)dx$
   
   (n) $\int x \sin^{-1}(x)dx$ (Skip this one)
   
   (o) $\int (r^2+r+1)e^r dr$
   
   (p) $\displaystyle\int \frac{1}{\sqrt{x+1}}dx$
   
   (q) $\displaystyle\int \frac{6x^3+9x+5}{x^4+3x^2-4}dx$
   
   (r) $\displaystyle\int \frac{1}{e^{3x}-e^x}dx$ (This one is a little tricky)
   
   (s) $\displaystyle\int \frac{x^2}{x^3+7}dx$
   
   (t) $\displaystyle\int_{-1}^2 x^2 e^{x^3} dx$
   
   (u) $\int \sin 3x \cos 5x dx$ (Hint: This requires one of those weird trig. formulas)
   
   (v) $\int \tan^3 x \sec^3 x dx$
   
   (w) $\int \sin^3 x \cos x dx$
   
   (x) $\int \tan^4 x \sec^2 x dx$

2. Calculate these integrals, too:

   (a) $\displaystyle\int x^3\sqrt{x^2+2}dx$ (This one is a little tricky)
   
   (b) $\displaystyle\int \frac{2x}{(x+1)(2x-2)}dx$
   
   (c) $\displaystyle\int \frac{x^3-2x^2+3x}{x^2-1}dx$

(d) $\int \frac{1}{x^3+x}dx$

3. Calculate $\int x^2e^{4x}dx$. Calculate the average value of $f(x) = x^2e^{4x}$ on the interval $[0, 2]$.

4. Find the length of the curve $y = \ln(\sec x)$ from $x = 0$ to $x = \frac{\pi}{4}$

5. Calculate the following integrals:

(a) $\int \frac{dx}{9-x^2}$

$$x = 3\sin\theta \qquad dx = 3\cos\theta d\theta$$

$$= \int \frac{3\cos\theta d\theta}{9-9\sin^2\theta}$$

$$= \int \frac{3\cos\theta d\theta}{9\cos^2\theta}$$

$$= \frac{1}{3}\int \frac{d\theta}{\cos\theta}$$

$$= \frac{1}{3}\int \sec\theta d\theta$$

$$= \frac{1}{3}\ln\mid\sec\theta+\tan\theta\mid+C$$

(b) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

$$= \int \frac{1}{3}\cdot\frac{dx}{\sqrt{1-\frac{x^2}{9}}}$$

$$= \frac{1}{3}\int \frac{dx}{\sqrt{1-(\frac{x}{3})^2}}$$

$$u = \frac{x}{3} \qquad du = \frac{1}{3}dx$$

$$= \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{3}\sin^{-1}\frac{x}{3}+C$$

(c) $\int \frac{dx}{\sqrt{-2x-x^2}}$

$$= \int \frac{dx}{\sqrt{(-x^2+2x)}}$$

$$= \int \frac{dx}{\sqrt{-(x^2+2x+1-1)}}$$

$$= \int \frac{dx}{\sqrt{-(x+1)^2+1}}$$

$$= \int \frac{dx}{\sqrt{1-(x+1)^2}}$$

$$u = x + 1 \qquad du = dx$$

$$= \int \frac{1}{\sqrt{1-u^2}} du$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1}(x+1) + C$$

(d) $\int \frac{2 - \cos x + \sin x}{\sin^2 x} dx$

$$= \int \frac{2}{\sin^2 x} - \frac{\cos x}{\sin^2 x} + \frac{\sin x}{\sin^2 x} dx$$

$$= \int 2\csc^2 x - \cot x \csc x + \csc x dx$$

$$= -2\cot x + \csc x - \ln \mid \csc x + \cot x \mid + C$$

(e) $\int \theta \cos(2\theta + 1) d\theta$

Use integration by parts.

$$u = \theta \qquad dv = \cos(2\theta+1)d\theta$$

$$du = d\theta \qquad v = \frac{1}{2}\sin(2\theta+1)$$

$$\int \theta \cos(2\theta+1)d\theta = \frac{1}{2}\theta \sin(2\theta+1) - \frac{1}{2}\int \sin(2\theta+1)d\theta$$

$$= \frac{1}{2}\theta\sin(2\theta+1) + \frac{1}{4}\cos(2\theta+1) + C$$

(f) $\int \frac{x^3}{x^2 - 2x + 1} dx$

Use polynomial long division to rewrite this as:

$$= \int x + 2 + \frac{3x-2}{x^2-2x+1} dx$$

$$= \frac{x^2}{2} + 2x + \int \frac{3x-2}{(x-1)^2}$$

$$= \frac{x^2}{2} + 2x + \int \frac{A}{x-1} + \frac{B}{(x-1)^2} dx$$

Solving for the coefficients:

$$A(x-1) + B = 3x - 2$$

When $x = 1$:

$$B = 3(1) - 2 = 1$$

$$Ax - A + 1 = 3x - 2$$

$$A = 3$$

So the integral is:

$$= \frac{x^2}{2} + 2x + \int \frac{3}{x-1} + \frac{1}{(x-1)^2} dx$$

$$= \frac{x^2}{2} + 2x + 3\ln \mid x - 1 \mid - \frac{1}{x-1} + C$$

(g) $\int \frac{dx}{\sqrt{1+\sqrt{x}}}$

$$u = 1+\sqrt{x} \qquad u-1=\sqrt{x} \qquad du = \frac{dx}{2\sqrt{x}}$$

$$2\sqrt{x}du = dx \qquad 2(u-1)du = dx$$

so the integral is;

$$= \int \frac{2(u-1)du}{\sqrt{u}}$$

$$= 2\int \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}du$$

$$= 2\left(\frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u}\right) + C$$

$$= \frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}} - 2\sqrt{1+\sqrt{x}} + C$$

(h) $\int \frac{2\sin\sqrt{x}dx}{\sqrt{x}\sec\sqrt{x}}$

$$u = \sqrt{x} \qquad du = \frac{1}{2\sqrt{x}}dx$$

$$= \int \frac{4\sin u du}{\sec u}$$

$$= 4\int \sin u \cos u du$$

$$w = \sin u \qquad dw = \cos u du$$

$$= 4\int w dw$$

$$= 4\frac{w^2}{2} + C$$

$$= 4\frac{\sin^2\sqrt{x}}{2} + C$$

$$= 2\sin^2\sqrt{x} + C$$

(i) $\int \frac{\sin 2x dx}{(1+\cos 2x)^2}$

$$u = 1+\cos 2x \qquad du = -2\sin 2x dx$$

$$-\frac{1}{2}dx = \sin 2x dx$$

$$= -\frac{1}{2}\int \frac{du}{u^2}$$

$$= -\frac{1}{2}\left(-\frac{1}{u}\right) + C$$

$$= \frac{\frac{1}{2}}{1+\cos 2x} + C$$

(j) $\int \frac{dy}{y^2 - 2y + 2}$

$$= \int \frac{dy}{y^2 - 2y + 1 - 1 + 2}$$
$$= \int \frac{dy}{(y^2 - 2y + 1) + 1}$$
$$= \int \frac{dy}{(y-1)^2 + 1}$$
$$u = y - 1 \qquad du = dy$$
$$= \int \frac{du}{u^2 + 1}$$
$$= \tan^{-1} u + C$$
$$= \tan^{-1}(y - 1) + C$$

(k) $\int \ln \sqrt{x - 1} dx$

$$= \int \frac{1}{2} \ln(x - 1) dx$$
$$= \frac{1}{2} \int \ln(x - 1) dx$$
$$u = \ln(x - 1) \qquad dv = dx$$
$$du = \frac{1}{x - 1} dx \qquad v = x$$
$$= x \ln(x - 1) - \int x \frac{1}{x - 1} dx$$
$$= x \ln(x - 1) - \int \frac{x - 1 + 1}{x - 1} dx$$
$$= x \ln(x - 1) - \int 1 + \frac{1}{x - 1} dx$$
$$= x \ln(x - 1) - (x + \ln \mid x - 1 \mid) + C$$
$$= x \ln(x - 1) - x - \ln \mid x - 1 \mid + C$$

(l) $\int \frac{x + 1}{x^2(x^2 + 1)} dx$

$$= \int \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} dx$$

Solving for the coefficients:

$$Ax(x^2 + 1) + B(x^2 + 1) + (Cx + D)x^2 = x + 1$$

If $x = 0$:

$$B = 1$$
$$Ax^3 + Ax + x^2 + 1 + Cx^3 + Dx^2 = x + 1$$
$$(A + C)x^3 + (D + 1)x^2 + Ax + 1 = x + 1$$
$$A = 1$$

So we have

$$(1+C)x^3 + (D+1)x^2 + 1x + 1 = x + 1$$
$$D + 1 = 0 \Rightarrow D = -1$$
$$1 + C = 0 \Rightarrow C = -1$$

So we have:

$$= \int \frac{1}{x} + \frac{1}{x^2} + \frac{-x-1}{x^2+1} dx$$
$$= \ln \mid x \mid - \frac{1}{x} + \int \frac{-x}{x^2+1} dx - \int \frac{1}{x^2+1} dx$$
$$= \ln \mid x \mid - \frac{1}{x} - \frac{1}{2} \ln(x^2+1) - \tan^{-1} x + C$$

(m) $\int x^3 e^{x^2} dx$

$$= \int x^2 x e^{x^2} dx$$
$$u = x^2 \qquad dv = x e^{x^2} dx$$
$$du = 2x dx \qquad v = \frac{1}{2} e^{x^2}$$
$$= \frac{1}{2} x^2 e^{x^2} - \int x e^{x^2} dx$$
$$= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C$$

(n) $\int \sin^2 x dx$

$$= \int \frac{1 - \cos 2x}{2} dx$$
$$= \frac{1}{2} x - \frac{1}{4} \sin(2x) + C$$

(o) $\int \frac{e^t dt}{1 + e^t}$

$$u = 1 + e^t \qquad du = e^t dt$$
$$= \int \frac{1}{u} du$$
$$= \ln \mid u \mid + C$$
$$= \ln \mid 1 + e^t \mid + C$$

(p) $\int \frac{\cot v dv}{\ln(\sin v)}$

$$u = \ln(\sin v) \qquad du = \frac{1}{\sin v} \cdot \cos v dv = \frac{\cos v}{\sin v} dv = \cot v dv$$
$$= \int \frac{1}{u} du$$
$$= \ln \mid u \mid + C$$
$$= \ln |\ln \sin v| + C$$

(q) $\int e^x \cos(2x)dx$

$$u = \cos(2x) \qquad dv = e^x dx$$

$$du = -2\sin(2x)dx \qquad v = e^x$$

$$\int e^x \cos(2x)dx = e^x \cos(2x) + 2\int e^x \sin(2x)dx$$

$$u = \sin(2x) \qquad dv = e^x dx$$

$$du = 2\cos(2x)dx \qquad v = e^x$$

$$\int e^x \cos(2x)dx = e^x \cos(2x) + 2\left(e^x \sin(2x) - 2\int e^x \cos(2x)dx\right)$$

$$\int e^x \cos(2x)dx = e^x \cos(2x) + 2e^x \sin(2x) - 4\int e^x \cos(2x)dx$$

$$5\int e^x \cos(2x)dx = e^x \cos(2x) + 2e^x \sin(2x) + C$$

$$\int e^x \cos(2x)dx = \frac{1}{5}e^x \cos(2x) + \frac{2}{5}e^x \sin(2x) + C$$

(r) $\int \frac{dx}{(x^2-1)^{\frac{3}{2}}}$

$$= \int \frac{dx}{(x-2)(x-1)}$$

$$= \int \frac{A}{x-2} + \frac{B}{x-1}dx$$

Solving for the coefficients we have

$$A(x-1) + B(x-2) = 1$$

If $x = 1$

$$B(-1) = 1$$

$$B = -1$$

If $x = 2$

$$A = 1$$

So the integral is:

$$= \int \frac{1}{x-2} + \frac{-1}{x-1}dx$$

$$= \ln \mid x-2 \mid - \ln \mid x-1 \mid + C$$

$$= \ln \left| \frac{x-2}{x-1} \right| + C$$

6. Determine if the following integrals converge or diverge. Give reasons for your answers. If the integral converges, find it's value if possible.

   (a) $\int_0^1 \ln x dx$

$$= \lim_{t \to 0} \int_t^1 \ln x dx$$

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x} dx \qquad v = x$$

$$= \lim_{t \to 0} \left( x \ln x - \int x \cdot \frac{1}{x} dx \right)$$

$$= \lim_{t \to 0} (x \ln x - x|_t^1$$

$$= \lim_{t \to 0} -1 - t \ln t + t$$

$$= -1 - \lim_{t \to 0} \frac{\ln t}{\frac{1}{t}}$$

$$=^{LH} -1 - \lim_{t \to 0} \frac{\frac{1}{t}}{-\frac{1}{t^2}}$$

$$= -1 + \lim_{t \to 0} \frac{t^2}{t}$$

$$= -1 - \lim_{t \to 0} t$$

$$= 1$$

   Therefore the integral converges.

   (b) $\int_3^5 \frac{1}{x-4} dx$

$$= \int_3^4 \frac{1}{x-4} dx + \int_4^5 \frac{1}{x-4} dx$$

   We will do the first integral first:

$$\int_3^4 \frac{1}{x-4} dx = \lim_{t \to 4^-} \int_3^t \frac{1}{x-4} dx$$

$$= \lim_{t \to 4^-} (\ln \mid x - 4 \mid|_3^t$$

$$= \lim_{t \to 4^-} \ln \mid t - 4 \mid - \ln \mid -1 \mid$$

$$= \lim_{t \to 4^-} \ln \mid t - 4 \mid -0$$

$$= -\infty$$

   Therefore, since one part of the integral diverges, the entire integral diverges.

(c) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

$$\begin{aligned}
&= \lim_{t\to 3^-} \int_0^t \frac{dx}{\sqrt{9-x^2}} \\
&= \lim_{t\to 3^-} \left(\frac{1}{3}\sin^{-1}\left(\frac{x}{3}\right)\right)_0^t \\
&= \lim_{t\to 3^-} \frac{1}{3}\sin^{-1}\left(\frac{t}{3}\right) - \frac{1}{3}\sin^{-1} 0 \\
&= \frac{1}{3}\frac{\pi}{2} - 0 \\
&= \frac{\pi}{6}
\end{aligned}$$

so the integral converges.

(d) $\int_0^\infty \frac{2dx}{x^2-2x}$

$$= \int_0^1 \frac{2dx}{x^2-2x} + \int_1^2 \frac{2dx}{x^2-2x} + \int_2^3 \frac{2dx}{x^2-2x} + \int_3^\infty \frac{2dx}{x^2-2x}$$

We will look at the last one:

$$\begin{aligned}
\int_3^\infty \frac{2dx}{x^2-2x} &= \lim_{t\to\infty} 2\int_3^t \frac{dx}{x(x-2)} \\
&= \lim_{t\to\infty} 2\int_3^t \frac{A}{x} + \frac{B}{x-2}dx
\end{aligned}$$

solving for the coefficients:

$$A(x-2) + Bx = 1$$

If $x = 0 \Rightarrow -2A = 1 \Rightarrow A = -\frac{1}{2}$
If $x = 2 \Rightarrow 2B = 1 \Rightarrow B = \frac{1}{2}$

$$\begin{aligned}
&= \lim_{t\to\infty} 2\int_3^t \frac{-\frac{1}{2}}{x} + \frac{\frac{1}{2}}{x-2}dx \\
&= \lim_{t\to\infty} 2\left(-\frac{1}{2}\ln|\,x\,| + \frac{1}{2}\ln|\,x-2\,|\right)_3^t \\
&= \lim_{t\to\infty} (-\ln|\,x\,| + \ln|\,x-2\,|)_3^t \\
&= \lim_{t\to\infty} \left(\ln\left|\frac{x-2}{x}\right|\right)_3^t \\
&= \lim_{t\to\infty} \ln\left|\frac{t-2}{t}\right| - \ln\frac{1}{3} \\
&= \ln 1 - \ln\frac{1}{3} = -\ln\frac{1}{3}
\end{aligned}$$

Let us look at

$$\begin{aligned}
\int_0^1 \frac{2dx}{x^2-2x} &= \lim_{t\to 0^+} \int_t^1 \frac{2dx}{x^2-2x} \\
&= \lim_{t\to 0^+} \int_t^1 \frac{-\frac{1}{2}}{x} + \frac{\frac{1}{2}}{x-2}dx
\end{aligned}$$

$$= \lim_{t\to 0^+} \left( \ln \left| \frac{x-2}{x} \right| \right)_t^1$$

$$= \lim_{t\to 0^+} \ln \left| \frac{-1}{1} \right| - \ln \left| \frac{t-2}{t} \right|$$

$$= 0 - \lim_{t\to 0^+} \ln \left| \frac{t}{t} - \frac{2}{t} \right|$$

$$= 0 - \infty = -\infty$$

So this integral diverges.

(e) $\int_1^\infty \frac{3x-1}{4x^3-x^2}$

$$= \lim_{t\to\infty} \int_1^t \frac{3x-1}{x^2(2x-1)} dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x-1} dx$$

solving for the coefficients:

$$Ax(2x-1) + B(2x-1) + Cx^2 = 3x - 1$$

If $x = 0 \Rightarrow -B = -1 \Rightarrow B = 1$
If $x = \frac{1}{2} \Rightarrow \frac{1}{4}C = \frac{3}{2} - 1 = \frac{1}{2} \Rightarrow C = 2$

$$2Ax^2 - Ax + 2Bx - B + Cx^2 = 3x - 1$$

$$(2A + C)x^2 - (2B - A)x - B = 3x - 1$$

$$(2A + 2)x^2 + (A - 2)x - 1 = 3x - 1$$

$$A - 2 = 3 \Rightarrow A = 5$$

$$= \lim_{t\to\infty} \int_1^t \frac{5}{x} + \frac{1}{x^2} + \frac{2}{2x-1} dx$$

$$= \lim_{t\to\infty} \left( 5 \ln \mid x \mid - \frac{1}{x} + \ln \mid 2x - 1 \mid \right)_1^t$$

$$= \lim_{t\to\infty} 5 \ln \mid t \mid - \frac{1}{t} + \ln \mid 2x - 1 \mid -0 + 1 + 0$$

$$= \infty$$

so the integral diverges.

(f) $\int_0^\infty x^2 e^{-x} dx$

$$= \lim_{t\to\infty} \int_0^t x^2 e^{-x} dx$$

$$u = x^2 \qquad dv = e^{-x} dx$$

$$du = 2x dx \qquad v = -e^{-x}$$

$$= \lim_{t\to\infty} -x^2 e^{-x} + 2 \int_0^t x e^{-x} dx$$

$$u = x \qquad dv = e^{-x} dx$$

$$du = dx \qquad v = -e^{-x}$$

$$= \lim_{t\to\infty} -x^2e^{-x} + 2\left(-xe^{-x} + \int_0^t e^{-x}dx\right)$$

$$= \lim_{t\to\infty} \left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right)_0^t$$

$$= \lim_{t\to\infty} \left(-t^2e^{-t} - 2te^{-t} - 2e^{-t}\right) - (-2) = 0 + 2 = 0$$

so the integral converges.

(g) $\int_1^\infty \frac{e^{-t}}{\sqrt{t}}dt$

$$0 < \frac{e^{-t}}{\sqrt{t}} \le e^{-t} \text{ for } t \ge 1$$

$$\int_1^\infty e^{-t} = \lim_{s\to\infty} -e^{-t} \mid_1^s$$

$$= \lim_{s\to\infty} -e^{-s} + \frac{1}{e}$$

$$= \frac{1}{e}$$

Since the larger integral converges, $\int_1^\infty \frac{e^{-t}}{\sqrt{t}}dt$ also converges, although we can't find it's value.

7. Calculate the following limits:

(a) $\lim_{x\to 0} \frac{x\sin x}{1-\cos x}$

$$=^{LH} \lim_{x\to 0} \frac{1\sin x + x\cos x}{\sin x}$$

$$=^{LH} \lim_{x\to 0} \frac{-x\sin x + \cos x + \cos x}{\cos x}$$

$$= \lim_{x\to 0} \frac{2\cos x - x\sin x}{\cos x} = \frac{2-0}{1} = 2$$

(b) $\lim_{x\to\infty} x^{\frac{1}{1-x}}$

$$f(x) = x^{\frac{1}{1-x}}$$

$$\ln f(x) = \frac{1}{1-x}\ln x = \frac{\ln x}{1-x}$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{1-x}$$

$$=^{LH} \lim_{x\to\infty} \frac{\frac{1}{x}}{-1}$$

$$= \lim_{x\to\infty} -\frac{1}{x} = 0$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^{\lim_{x\to\infty} \ln f(x)} = e^0 = 1$$

(c) $\lim_{t\to 0} \frac{\tan 3t}{\tan 5t}$

$$=^{LH} \lim_{t\to 0} \frac{3\sec^2 3t}{5\sec^2 5t} = \frac{3}{5}$$