Math 143
Spring, 2003
Exam 1
Chapters 4, 5, and 6

Part I - Definitions and Examples

1. (5 points) State the definition of an integrable function.

   Solution: A function $f(x)$ is integrable on $[a, b]$ if $\int_a^b f(x)dx$ exists, i.e. for any partition p of $[a, b]$ let the numbers c_i be chosen arbitratily in the subintervals $[x_{i-1}, x_i]$. If there exists a number I so that $\lim_{||P||\to 0} \Sigma_{i=1}^n f(c_i)\Delta x_i = I$ no matter how P or the c_k's are chosen, then f is integrable on $[a, b]$ and I is the definite integral of f over $[a, b]$.

2. (5 points) State the Fundamental Theorem of Calculus, part II. Include all necessary hypotheses.

   Solution: If f is continuous at every point of $[a, b]$ and if F is any antiderivative of f on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

3. (5 points) Explain why this part of the Fundamental Theorem of Calculus is important. (i.e., what makes it fundamental?)

   Solution: This is important because the fundamental theorem relates integral and differential calculus. Also this gives us a means for evaluating definite integrals when we know an antiderivative of the function - we do not have to use Riemann sums and limits, but can use an antiderivative to calculate the area under the curve.

4. (4 points) State the definition of $\ln x$ which uses calculus.

   Solution:
   $$\ln x = \int_1^x \frac{1}{t}dt \qquad \text{for } x > 0$$

   Part II - Calculations

5. (6 points) Find $\frac{d}{dx}\int_{\sin x}^0 \frac{1}{1+t^2}dt$

   Solution:
   $$\begin{aligned} &= \frac{d}{dx}\left(-\int_0^{\sin x} \frac{1}{1+t^2}dt\right) \\ &= -\frac{1}{1+\sin^2 x} \cdot \cos x \\ &= -\frac{\cos x}{1+\sin^2 x} \end{aligned}$$

   by FTC I.

6. (8 points) Find $\int x^e + \pi^e + e^\pi + e^{\pi x}dx$

   Solution:
   $$= \frac{1}{e+1}x^{e+1} + \pi^e x + e^\pi x + \frac{1}{\pi}e^{\pi x} + C$$

7. (6 points) Find $\int \sin x + \frac{1}{1+x^2} + \sec^2 x dx$

Solution:

$$= -\cos x + \tan^1 x + \tan x + C$$

8. (6 points) Find $\int_2^6 \mid x - 3 \mid dx$

Solution: The area under the curve makes two triangles, so the area is the sum of the areas of the triangles:

$$A = \frac{1}{2}(1)(1) + \frac{1}{2}(3)(3) = \frac{1}{2} + \frac{9}{2} = \frac{10}{2} = 5$$

9. (6 points) $\int 10^{2\theta} d\theta$

Solution:

$$\begin{aligned} &= \frac{1}{\ln 10}\frac{1}{2}10^{2\theta} + C \\ &= \frac{1}{2\ln 10}10^{2\theta} + C \end{aligned}$$

10. (6 points) $\int \sin^3 x \cos x dx$

Solution:

$$u = \sin x \qquad du = \cos x dx$$

$$\begin{aligned} &= \int u^3 du \\ &= \frac{1}{4}u^4 + C \\ &= \frac{1}{4}\sin^4 x + C \end{aligned}$$

11. (6 points) $\int x \tan(x^2) dx$

Solution:

$$u = x^2 \qquad du = 2x dx$$

$$= \frac{1}{2}\int \tan u du = \frac{1}{2}\ln \mid \sec x^2 \mid + C$$

12. (8 points) Find the area of the region bounded by $y = x^3$ and $x = y^3$

Solution: First we solve the second equation for y, and so we are looking for the area of the region bounded by $y = x^3$ and $y = x^{\frac{1}{3}}$. These graphs intersect at $x = -1$, $x = 0$, and $x = 1$. Drawing the picture, we can tell which function is "above" the other to get the following integrals:

$$\begin{aligned} A &= \int_{-1}^{0} (x^3 - x^{\frac{1}{3}}) dx + \int_0^1 x^{\frac{1}{3}} - x^3 dx \\ &= \left(\frac{1}{4}x^4 - \frac{4}{3}x^{\frac{4}{3}}\right)_{-1}^{0} + \left(\frac{3}{4}x^{\frac{4}{3}} - \frac{1}{4}x^4\right)_0^1 \\ &= -\frac{1}{4} + \frac{3}{4} + \frac{3}{4} - \frac{1}{4} \\ &= \frac{6}{4} - \frac{2}{4} = 1 \end{aligned}$$

13. Solve the following differential equations:

(a) (10 points) $e^x \frac{dy}{dx} + 2e^x y = 1 \qquad y(0) = 5$

Solution:

$$\frac{dy}{dx} + 2y = e^{-x}$$

$$\mu(x) = e^{\int 2dx} = e^{2x}$$

$$y = \frac{1}{\mu(x)} \int \mu(x)g(x)dx$$

$$= \frac{1}{e^{2x}} \int e^{2x}e^{-x}dx$$

$$= \frac{1}{e^{2x}} \int e^{x}dx$$

$$= \frac{1}{e^{2x}} (e^x + C)$$

Now use the initial condition:

$$y(0) = 5$$

$$5 = \frac{1}{1}(1 + C)$$

$$5 = 1 + C$$

$$4 = C$$

So,

$$y = \frac{1}{e^{2x}} (e^x + 4)$$

(b) (8 points) $\frac{dy}{dx} = \frac{\sin y}{\cos x}$

Solution:

$$\frac{1}{\sin y}dy = \frac{1}{\cos x}dx$$

$$\csc y dy = \sec x dx$$

$$-\ln \mid \csc y + \cot y \mid = \ln \mid \sec x + \tan x \mid + C$$

14. (8 points) Use a Riemann sum with $n = 4$ subintervals, and the right endpoint of each subinterval, to approximate the area under the curve $y = x^2 + 1$ on the interval $[0, 2]$. Is this an overestimate or an underestimate? Why? (A picture would work for this part.)

Solution:

Interval	Right endpoint	function value at right endpoint	Δx	Area
$0 - 0.5$	.5	1.25	.5	.625
$0.1 - 1.0$	1	2	.5	1
$1.0 - 1.5$	1.5	3.25	.5	1.625
$1.5 - 2.0$	2	5	.5	2.5

Therefore the total area is $.625 + 1 + 1.625 + 2.5 = 5.75$. Looking at the graph, we see that the right hand endpoints overestimate the area (they are the higher end of each region).

15. (5 points) Set up but do not integrate the integral to compute the volume of the solid generated by rotating the region bounded by the curves $y = x$ and $y = x^3$ about the line $x = 3$.

    Solution: We solve the second equation for x to get that $x = y^{\frac{1}{3}}$. We are rotating about the line $x = 3$, which is like rotating about a translated $y-$axis. We will use discs, and need to end up with everything in terms of y. Notice that there are two radii, and each is $3-$(the function value there).

$$V = \pi \int_0^1 (3-y)^2 - (3-\sqrt[3]{y})^2 dy$$

16. (5 points) Set up but do not integrate the integral to compute the volume of the solid generated rotating the region bounded by the curves $y = 8x - x^4$, $x = 0$, and $y = 7$ about the $y-$axis.

    Solution: When $y = 7$, $x = 1$, and we will do this volume by cylindrical shells:

$$V = \int_0^1 2\pi x(7 - (8x - x^4))dx$$

    Notice that the height is $7-$(the function value).

17. (10 points) Find the length of the curve given by $x = \frac{t^2}{2}$, $y = \frac{(2t+1)^{\frac{3}{2}}}{3}$ from $t = 0$ to $t = 3$.

    Solution: Recall that the formula for the arc length of a parametric equation is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\frac{dx}{dt} = t$$

$$\frac{dy}{dt} = \frac{1}{3} \cdot \frac{3}{2} (2t+1)^{\frac{1}{2}} \cdot 2 = \sqrt{2t+1}$$

    So the formula for arc length is:

$$\begin{aligned} L &= \int_0^3 \sqrt{t^2 + \sqrt{2t+1}^2} dt \\ &= \int_0^3 \sqrt{t^2 + 2t + 1} dt \\ &= \int_0^3 \sqrt{(t+1)^2} dt \\ &= \int_0^3 t + 1 dt \\ &= \frac{1}{2}t^2 + t \mid_0^3 \end{aligned}$$

$$= \frac{1}{2}(9) + 3 = 4.5 + 3 = 7.5$$

18. (10 points) Solve the differential equation $\sqrt{x}\frac{dy}{dx} = e^{y+\sqrt{x}}$.

Solution:

$$\sqrt{x}\frac{dy}{dx} = e^y e^{\sqrt{x}}$$
$$e^{-y}dy = \frac{1}{\sqrt{x}}e^{\sqrt{x}}dx$$
$$\int e^{-y}dy = \int \frac{1}{\sqrt{x}}e^{\sqrt{x}}dx$$
$$-e^{-y} = 2e^{\sqrt{x}} + C$$
$$2e^{\sqrt{x}} + e^{-y} = C$$

Part III - Applications

19. (10 points) A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 24 cm?

Solution: We use Hooke's Law: $F = kx$ where k is the spring constant. Remembering that Hooke's law requires the measurements to be in meters.

$$F = 30 = k(.03)$$
$$k = 1000$$

Work is the integral of force when force changes, so;

$$W = \int_0^{.12} 1000x dx = 500x^2 \mid_0^{.12} = 7.2J$$