Here are some things to look at as you prepare for the final exam. Remember to look at review sheets for other exams, previous exams, and quizzes.

1. Know the statement of the following theorems and definitions:
   (a) The definition of the definite integral of f on $[a, b]$
   (b) The Mean Value Theorem for Definite Integrals
   (c) The Fundamental Theorem of Calculus, Parts I and II
   (d) The definition of $\ln x$ using calculus
   (e) The Sandwich Theorem for Sequences
   (f) Monotonic Sequence Theorem
   (g) $n-$th term test
   (h) The Integral Test
   (i) The Comparison Test for Series
   (j) The Ratio Test
   (k) The Root Test
   (l) Leibniz's theorem / Alternating Series Test
   (m) The Absolute Convergence Test
   (n) The Convergence Theorem for Power Series

2. Be able to do the following integrals:
   (a) $\displaystyle\int \frac{dx}{9 - x^2}$

$$x = 3\sin\theta \qquad\qquad dx = 3\cos\theta d\theta$$

$$= \int \frac{3\cos\theta d\theta}{9 - 9\sin^2\theta}$$
$$= \int \frac{3\cos\theta d\theta}{9\cos^2\theta}$$
$$= \frac{1}{3}\int \frac{d\theta}{\cos\theta}$$
$$= \frac{1}{3}\int \sec\theta d\theta$$
$$= \frac{1}{3}\ln|\sec\theta + \tan\theta| + C$$

(b) $\int \frac{dx}{\sqrt{9-x^2}}$

$$= \int \frac{1}{3} \cdot \frac{dx}{\sqrt{1-\frac{x^2}{9}}}$$

$$= \frac{1}{3} \int \frac{dx}{\sqrt{1-\left(\frac{x}{3}\right)^2}}$$

$$u = \frac{x}{3} \qquad du = \frac{1}{3}dx$$

$$= \int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1} \frac{x}{3} + C$$

(c) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

$$u = \sqrt{x} \qquad du = \frac{1}{2\sqrt{x}}dx$$

$$= \int 2\cos u du$$

$$= 2\sin u + C$$

$$= 2\sin\sqrt{x} + C$$

(d) $\int \frac{dx}{\sqrt{-2x-x^2}}$

$$= \int \frac{dx}{\sqrt{-(x^2+2x)}}$$

$$= \int \frac{dx}{\sqrt{-(x^2+2x+1-1)}}$$

$$= \int \frac{dx}{\sqrt{-\left((x+1)^2-1\right)}}$$

$$= \int \frac{dx}{\sqrt{-(x+1)^2+1}}$$

$$= \int \frac{dx}{\sqrt{1-(x+1)^2}}$$

$$u = x+1 \qquad du = dx$$

$$= \int \frac{1}{\sqrt{1-u^2}}du$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1}(x+1) + C$$

(e) $\int \frac{2 - \cos x + \sin x}{\sin^2 x} dx$

$$= \int \frac{2}{\sin^2 x} - \frac{\cos x}{\sin^2 x} + \frac{\sin x}{\sin^2 x} dx$$

$$= \int 2\csc^2 x - \cot x \csc x + \csc x dx$$

$$= -2\cot x + \csc x - \ln \mid \csc x + \cot x \mid + C$$

(f) $\int \theta \cos(2\theta + 1) d\theta$

Use integration by parts.

$$u = \theta \qquad dv = \cos(2\theta + 1) d\theta$$

$$du = d\theta \qquad v = \frac{1}{2}\sin(2\theta + 1)$$

$$\int \theta \cos(2\theta + 1) d\theta = \frac{1}{2}\theta \sin(2\theta + 1) - \frac{1}{2}\int \sin(2\theta + 1) d\theta$$

$$= \frac{1}{2}\theta \sin(2\theta + 1) + \frac{1}{4}\cos(2\theta + 1) + C$$

(g) $\int \frac{x^3}{x^2 - 2x + 1} dx$

Use polynomial long division to rewrite this as:

$$= \int x + 2 + \frac{3x - 2}{x^2 - 2x + 1} dx$$

$$= \frac{x^2}{2} + 2x + \int \frac{3x - 2}{(x - 1)^2}$$

$$= \frac{x^2}{2} + 2x + \int \frac{A}{x - 1} + \frac{B}{(x - 1)^2} dx$$

Solving for the coefficients:

$$A(x - 1) + B = 3x - 2$$

When $x = 1$:

$$B = 3(1) - 2 = 1$$

$$Ax - A + 1 = 3x - 2$$

$$A = 3$$

So the integral is:

$$= \frac{x^2}{2} + 2x + \int \frac{3}{x - 1} + \frac{1}{(x - 1)^2} dx$$

$$= \frac{x^2}{2} + 2x + 3\ln \mid x - 1 \mid - \frac{1}{x - 1} + C$$

(h) $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{x}}}$

$$u = 1+\sqrt{x} \qquad u-1 = \sqrt{x} \qquad du = \frac{dx}{2\sqrt{x}}$$

$$2\sqrt{x}du = dx \qquad 2(u-1)du = dx$$

so the integral is;

$$= \int \frac{2(u-1)du}{\sqrt{u}}$$

$$= 2\int \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}du$$

$$= 2\left(\frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u}\right) + C$$

$$= \frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}} - 2\sqrt{1+\sqrt{x}} + C$$

(i) $\displaystyle\int \frac{2\sin\sqrt{x}dx}{\sqrt{x}\sec\sqrt{x}}$

$$u = \sqrt{x} \qquad du = \frac{1}{2\sqrt{x}}dx$$

$$= \int \frac{4\sin u du}{\sec u}$$

$$= 4\int \sin u \cos u du$$

$$w = \sin u \qquad dw = \cos u du$$

$$= 4\int w dw$$

$$= 4\frac{w^2}{2} + C$$

$$= 4\frac{\sin^2\sqrt{x}}{2} + C$$

$$= 2\sin^2\sqrt{x} + C$$

(j) $\displaystyle\int \frac{\sin 2x dx}{(1+\cos 2x)^2}$

$$u = 1+\cos 2x \qquad du = -2\sin 2x dx$$

$$-\frac{1}{2}dx = \sin 2x dx$$

$$= -\frac{1}{2}\int \frac{du}{u^2}$$

$$= -\frac{1}{2}\left(-\frac{1}{u}\right) + C$$

$$= \frac{\frac{1}{2}}{1+\cos 2x} + C$$

(k) $\int \frac{dy}{y^2 - 2y + 2}$

$$= \int \frac{dy}{y^2 - 2y + 1 - 1 + 2}$$

$$= \int \frac{dy}{(y^2 - 2y + 1) + 1}$$

$$= \int \frac{dy}{(y-1)^2 + 1}$$

$$u = y - 1 \qquad du = dy$$

$$= \int \frac{du}{u^2 + 1}$$

$$= \tan^{-1} u + C$$

$$= \tan^{-1}(y - 1) + C$$

(l) $\int \ln \sqrt{x-1} dx$

$$= \int \frac{1}{2} \ln(x-1) dx$$

$$= \frac{1}{2} \int \ln(x-1) dx$$

$$u = \ln(x-1) \qquad dv = dx$$

$$du = \frac{1}{x-1} dx \qquad v = x$$

$$= x \ln(x-1) - \int x \frac{1}{x-1} dx$$

$$= x \ln(x-1) - \int \frac{x-1+1}{x-1} dx$$

$$= x \ln(x-1) - \int 1 + \frac{1}{x-1} dx$$

$$= x \ln(x-1) - (x + \ln \mid x-1 \mid) + C$$

$$= x \ln(x-1) - x - \ln \mid x-1 \mid + C$$

(m) $\int \frac{x+1}{x^2(x^2+1)} dx$

$$= \int \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} dx$$

Solving for the coefficients:

$$Ax(x^2 + 1) + B(x^2 + 1) + (Cx + D)x^2 = x + 1$$

If $x = 0$:

$$B = 1$$

$$Ax^3 + Ax + x^2 + 1 + Cx^3 + Dx^2 = x + 1$$

$$(A+C)x^3+(D+1)x^2+Ax+1=x+1$$
$$A=1$$

So we have

$$(1+C)x^3+(D+1)x^2+1x+1=x+1$$
$$D+1=0\Rightarrow D=-1$$
$$1+C=0\Rightarrow C=-1$$

So we have:

$$=\int \frac{1}{x}+\frac{1}{x^2}+\frac{-x-1}{x^2+1}dx$$
$$=\ln\mid x\mid -\frac{1}{x}+\int \frac{-x}{x^2+1}dx-\int \frac{1}{x^2+1}dx$$
$$=\ln\mid x\mid -\frac{1}{x}-\frac{1}{2}\ln(x^2+1)-\tan^{-1}x+C$$

(n) $\int x^3 e^{x^2}dx$

$$=\int x^2 x e^{x^2}dx$$

$$u=x^2 \qquad dv=xe^{x^2}dx$$
$$du=2xdx \qquad v=\frac{1}{2}e^{x^2}$$

$$=\frac{1}{2}x^2e^{x^2}-\int xe^{x^2}dx$$
$$=\frac{1}{2}x^2e^{x^2}-\frac{1}{2}e^{x^2}+C$$

(o) $\int \sin^2 x\ dx$

$$=\int \frac{1-\cos 2x}{2}dx$$
$$=\frac{1}{2}x-\frac{1}{4}\sin(2x)+C$$

(p) $\int \frac{\cos(\sin^{-1}x)}{\sqrt{1-x^2}}dx$

$$u=\sin^{-1}x \qquad du=\frac{1}{\sqrt{1-x^2}}dx$$

$$=\int \cos u du$$
$$=\sin(\sin^{-1}x)+C$$
$$=x+C$$

(q) $\displaystyle\int \frac{e^t dt}{1+e^t}$

$$u = 1 + e^t \qquad\qquad du = e^t dt$$
$$= \int \frac{1}{u} du$$
$$= \ln \mid u \mid + C$$
$$= \ln \mid 1 + e^t \mid + C$$

(r) $\displaystyle\int \frac{\cot v \ dv}{\ln(\sin v)}$

$$u = \ln(\sin v) \qquad\qquad du = \frac{1}{\sin v} \cdot \cos v \ dv \qquad\qquad dv = \frac{\cos v}{\sin v} dv = \cot v \ dv$$
$$= \int \frac{1}{u} du$$
$$= \ln \mid u \mid + C$$
$$= \ln |\ln \sin v| + C$$

(s) $\displaystyle\int (27)^{3\theta+1} \, d\theta$

$$u = 3\theta + 1 \qquad\qquad du = 3 d\theta$$
$$= \int \frac{1}{3} (27)^u$$
$$= \frac{1}{3} \frac{1}{\ln 27} 27^u + C$$
$$= \frac{1}{3 \ln 27} 27^{3\theta+1} + C$$

(t) $\displaystyle\int e^x \cos(2x) dx$

$$u = \cos(2x) \qquad\qquad dv = e^x dx$$
$$du = -2\sin(2x) dx \qquad\qquad v = e^x$$
$$\int e^x \cos(2x) dx = e^x \cos(2x) + 2 \int e^x \sin(2x) dx$$
$$u = \sin(2x) \qquad\qquad dv = e^x dx$$
$$du = 2\cos(2x) dx \qquad\qquad v = e^x$$
$$\int e^x \cos(2x) dx = e^x \cos(2x) + 2 \left(e^x \sin(2x) - 2 \int e^x \cos(2x) dx \right)$$
$$\int e^x \cos(2x) dx = e^x \cos(2x) + 2e^x \sin(2x) - 4 \int e^x \cos(2x) dx$$
$$5 \int e^x \cos(2x) dx = e^x \cos(2x) + 2e^x \sin(2x) + C$$
$$\int e^x \cos(2x) dx = \frac{1}{5} e^x \cos(2x) + \frac{2}{5} e^x \sin(2x) + C$$

(u) $\int \frac{dx}{x^2 - 3x + 2}$

$$= \int \frac{dx}{(x-2)(x-1)}$$
$$= \int \frac{A}{x-2} + \frac{B}{x-1} dx$$

Solving for the coefficients we have

$$A(x-1) + B(x-2) = 1$$

If $x = 1$

$$B(-1) = 1$$
$$B = -1$$

If $x = 2$

$$A = 1$$

So the integral is:

$$= \int \frac{1}{x-2} + \frac{-1}{x-1} dx$$
$$= \ln \mid x-2 \mid - \ln \mid x-1 \mid + C$$
$$= \ln \left| \frac{x-2}{x-1} \right| + C$$

(v) $\int \frac{dx}{(x^2 - 1)^{\frac{3}{2}}}$

$$x = \sec\theta \qquad\qquad dx = \sec\theta\tan\theta d\theta$$

$$= \int \frac{\sec\theta\tan\theta d\theta}{(\sec^2\theta - 1)^{\frac{3}{2}}}$$
$$= \int \frac{\sec\theta\tan\theta d\theta}{(\tan^2\theta)^{\frac{3}{2}}}$$
$$= \int \frac{\sec\theta\tan\theta d\theta}{(\tan\theta)^3}$$
$$= \int \frac{\sec\theta}{\tan^2\theta} d\theta$$
$$= \int \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} d\theta$$
$$= \int \frac{\cos\theta}{\sin^2\theta} d\theta$$
$$= \int \cot\theta\csc\theta d\theta$$
$$= -\csc\theta + C$$
$$= -\frac{x}{\sqrt{x^2 - 1}} + C$$

3. Determine if the following integrals converge or diverge. Give reasons for your answers. If the integral converges, find it's value if possible.

(a) $\int_0^1 \ln x dx$

$$= \lim_{t\to 0} \int_t^1 \ln x dx$$

$$u = \ln x \qquad\qquad dv = dx$$

$$du = \frac{1}{x}dx \qquad\qquad v = x$$

$$= \lim_{t\to 0} \left(x \ln x - \int x \cdot \frac{1}{x} dx \right)$$

$$= \lim_{t\to 0} (x \ln x - x|_t^1$$

$$= \lim_{t\to 0} -1 - t \ln t + t$$

$$= -1 - \lim_{t\to 0} \frac{\ln t}{\frac{1}{t}}$$

$$=^{LH} -1 - \lim_{t\to 0} \frac{\frac{1}{t}}{-\frac{1}{t^2}}$$

$$= -1 + \lim_{t\to 0} \frac{t^2}{t}$$

$$= -1 - \lim_{t\to 0} t$$

$$= 1$$

Therefore the integral converges.

(b) $\int_3^5 \frac{1}{x-4} dx$

$$= \int_3^4 \frac{1}{x-4} dx + \int_4^5 \frac{1}{x-4} dx$$

We will do the first integral first:

$$\int_3^4 \frac{1}{x-4} dx = \lim_{t\to 4^-} \int_3^t \frac{1}{x-4} dx$$

$$= \lim_{t\to 4^-} (\ln \mid x-4 \mid|_3^t$$

$$= \lim_{t\to 4^-} \ln \mid t-4 \mid - \ln \mid -1 \mid$$

$$= \lim_{t\to 4^-} \ln \mid t-4 \mid -0$$

$$= -\infty$$

Therefore, since one part of the integral diverges, the entire integral diverges.

(c) $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

$$= \lim_{t\to 3^-} \int_0^t \frac{dx}{\sqrt{9-x^2}}$$
$$= \lim_{t\to 3^-} \left(\frac{1}{3}\sin^{-1}\left(\frac{x}{3}\right)\right)_0^t$$
$$= \lim_{t\to 3^-} \frac{1}{3}\sin^{-1}\left(\frac{t}{3}\right) - \frac{1}{3}\sin^{-1} 0$$
$$= \frac{1}{3}\frac{\pi}{2} - 0$$
$$= \frac{\pi}{6}$$

so the integral converges.

(d) $\displaystyle\int_0^\infty \frac{2dx}{x^2-2x}$

$$= \int_0^1 \frac{2dx}{x^2-2x} + \int_1^2 \frac{2dx}{x^2-2x} + \int_2^3 \frac{2dx}{x^2-2x} + \int_3^\infty \frac{2dx}{x^2-2x}$$

We will look at the last one:

$$\int_3^\infty \frac{2dx}{x^2-2x} = \lim_{t\to\infty} 2\int_3^t \frac{dx}{x(x-2)}$$
$$= \lim_{t\to\infty} 2\int_3^t \frac{A}{x} + \frac{B}{x-2}dx$$

solving for the coefficients:

$$A(x-2) + Bx = 1$$

If $x = 0 \Rightarrow -2A = 1 \Rightarrow A = -\frac{1}{2}$
If $x = 2 \Rightarrow 2B = 1 \Rightarrow B = \frac{1}{2}$

$$= \lim_{t\to\infty} 2\int_3^t \frac{-\frac{1}{2}}{x} + \frac{\frac{1}{2}}{x-2}dx$$
$$= \lim_{t\to\infty} 2\left(-\frac{1}{2}\ln\mid x\mid + \frac{1}{2}\ln\mid x-2\mid\right)_3^t$$
$$= \lim_{t\to\infty} (-\ln\mid x\mid + \ln\mid x-2\mid)_3^t$$
$$= \lim_{t\to\infty} \left(\ln\left|\frac{x-2}{x}\right|\right)_3^t$$
$$= \lim_{t\to\infty} \ln\left|\frac{t-2}{t}\right| - \ln\frac{1}{3}$$
$$= \ln 1 - \ln\frac{1}{3} = -\ln\frac{1}{3}$$

Let us look at

$$\int_0^1 \frac{2dx}{x^2-2x} = \lim_{t\to 0^+} \int_t^1 \frac{2dx}{x^2-2x}$$

$$= \lim_{t\to 0^+} \int_t^1 \frac{-\frac{1}{2}}{x} + \frac{\frac{1}{2}}{x-2} dx$$

$$= \lim_{t\to 0^+} \left(\ln \left| \frac{x-2}{x} \right| \right)_t^1$$

$$= \lim_{t\to 0^+} \ln \left| \frac{-1}{1} \right| - \ln \left| \frac{t-2}{t} \right|$$

$$= 0 - \lim_{t\to 0^+} \ln \left| \frac{t}{t} - \frac{2}{t} \right|$$

$$= 0 - \infty = -\infty$$

So this integral diverges.

(e) $\int_1^\infty \frac{3x-1}{4x^3-x^2}$

$$= \lim_{t\to\infty} \int_1^t \frac{3x-1}{x^2(4x-1)} dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{A}{x} + \frac{B}{x^2} + \frac{C}{4x-1} dx$$

solving for the coefficients:

$$Ax(4x-1) + B(4x-1) + Cx^2 = 3x-1$$

If $x = 0 \Rightarrow -B = -1 \Rightarrow B = 1$
If $x = \frac{1}{2} \Rightarrow \frac{1}{4}C = \frac{3}{2} - 1 = \frac{1}{2} \Rightarrow C = 4$

$$4Ax^2 - Ax + 4Bx - B + Cx^2 = 3x - 1$$

$$(4A + C)x^2 + (4B - A)x - B = 3x - 1$$

$$(2A + 4)x^2 + (4 - A)x - 1 = 3x - 1$$

$$4 - A = 3 \Rightarrow A = 1$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x} + \frac{1}{x^2} + \frac{4}{4x-1} dx$$

$$= \lim_{t\to\infty} \left(\ln \mid x \mid - \frac{1}{x} + \ln \mid 4x - 1 \mid \right)_1^t$$

$$= \lim_{t\to\infty} \left(\ln \mid t \mid - \frac{1}{t} + \ln \mid 4x - 1 \mid - 0 + 1 + 0 \right)$$

$$= \infty$$

so the integral diverges.

(f) $\int_0^\infty x^2 e^{-x} dx$

$$= \lim_{t\to\infty} \int_0^t x^2 e^{-x} dx$$

$$u = x^2 \qquad dv = e^{-x}dx$$

$$du = 2xdx \qquad v = -e^{-x}$$

$$= \lim_{t\to\infty} -x^2e^{-x} + 2\int_0^t xe^{-x}dx$$

$$u = x \qquad dv = e^{-x}dx$$

$$du = dx \qquad v = -e^{-x}$$

$$= \lim_{t\to\infty} -x^2e^{-x} + 2\left(-xe^{-x} + \int_0^t e^{-x}dx\right)$$

$$= \lim_{t\to\infty} \left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right)_0^t$$

$$= \lim_{t\to\infty} \left(-t^2e^{-t} - 2te^{-t} - 2e^{-t}\right) - (-2) = 0 + 2 = 0$$

so the integral converges.

(g) $\int_1^\infty \frac{e^{-t}}{\sqrt{t}} dt$

$$0 < \frac{e^{-t}}{\sqrt{t}} \leq e^{-t} \text{ for } t \geq 1$$

$$\int_1^\infty e^{-t} = \lim_{s\to\infty} -e^{-t} |_1^s$$

$$= \lim_{s\to\infty} -e^{-s} + \frac{1}{e}$$

$$= \frac{1}{e}$$

Since the larger integral converges, $\int_1^\infty \frac{e^{-t}}{\sqrt{t}} dt$ also converges, although we can't find it's value.

4. Calculate the following limits:

(a) $\lim_{x\to 0} \frac{x\sin x}{1-\cos x}$

$$=^{LH} \lim_{x\to 0} \frac{1\sin x + x\cos x}{\sin x}$$

$$=^{LH} \lim_{x\to 0} \frac{-x\sin x + \cos x + \cos x}{\cos x}$$

$$= \lim_{x\to 0} \frac{2\cos x - x\sin x}{\cos x} = \frac{2-0}{1} = 2$$

(b) $\lim_{x\to\infty} x^{\frac{1}{1-x}}$

$$e^{\ln f(x)} = f(x)$$

So examine just the exponent and then plug back in:

$$f(x) = x^{\frac{1}{1-x}}$$

$$\ln f(x) = \frac{1}{1-x}\ln x = \frac{\ln x}{1-x}$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{1-x}$$

$$=^{LH} \lim_{x\to\infty} \frac{\frac{1}{x}}{-1}$$

$$= \lim_{x\to\infty} -\frac{1}{x} = 0$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^{\lim_{x\to\infty} \ln f(x)} = e^0 = 1$$

(c) $\lim_{t\to 0} \frac{\tan 3t}{\tan 5t}$

$$=^{LH} \lim_{t\to 0} \frac{3\sec^2 3t}{5\sec^2 5t} = \frac{3}{5}$$

5. Solve the following differential equations

(a) $\frac{dy}{dx} = -\frac{y\ln y}{1+x^2}$ where $y(0) = e^2$

$$\frac{1}{y\ln y}dy = -\frac{1}{1+x^2}dx$$

$$\ln(\ln y) = -\tan^{-1} x + C$$

$$y(0) = e^2$$

$$\ln(\ln(e^2)) = \tan^{-1}(0) + C$$

$$\ln 2 = C$$

$$\ln(\ln y) = -\tan^{-1} x + \ln 2$$

(b) $\frac{dy}{dx} + \left(\frac{2}{x+1}\right) y = \frac{x}{x+1}$

$$\mu(x) = e^{\int \frac{2}{x+1}} = e^{2\ln|x+1|} = e^{\ln|x+1|^2} = (x+1)^2$$

$$y = \frac{1}{\mu(x)} \int \mu(x) g(x) dx$$

$$= \frac{1}{(x+1)^2} \int (x+1)^2 \frac{x}{x+1} dx$$

$$= \frac{1}{(x+1)^2} \int x(x+1) dx$$

$$= \frac{1}{(x+1)^2} \int x^2 + x dx$$

$$= \frac{1}{(x+1)^2} \left(\frac{x^3}{3} + \frac{x^2}{2} + C \right)$$

(c) $x\frac{dy}{dx} + 2y = x^2 + 1$

$$\frac{dy}{dx} + \frac{2}{x}y = x + \frac{1}{x}$$

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{\ln x^2} = x^2$$

$$y = \frac{1}{x^2} \int x^2 \left(x + \frac{1}{x} \right) dx$$

$$= \frac{1}{x^2} \int x^3 + x dx$$

$$= \frac{1}{x^2} \left(\frac{x^4}{4} + \frac{x^2}{2} + C \right)$$

(d) $xy' + y = x \cos x$

$$y' + \frac{1}{x}y = \cos x$$

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$y = \frac{1}{x} \int x \cos x dx$$

To do the integral, let

$$u = x \qquad dv = \cos x dx$$

$$du = dx \qquad v = \sin x$$

$$y = \frac{1}{x} \left(x \sin x - \int \sin x dx \right)$$

$$y = \frac{1}{x} (x \sin x + \cos x + C)$$

6. Find the volume of the solid generated by revolving the region bounded by the $x-$axis, the curve$y = 3x^4$ and the lines $x = 1$ and $x = -1$ about

(a) the $x-$axis

$$V = \int_{-1}^{1} \pi (3x^4)^2 dx$$

$$= \pi \int_{-1}^{1} 9x^8 dx$$

$$= \pi \left(x^9 \right)_{-1}^{1}$$

$$= \pi (1 - (-1)) = 2\pi$$

(b) the $y-$axis

$$V = \int_0^1 2\pi x(3x^4)dx$$
$$= 6pi \int_0^1 x^5 dx$$
$$= \pi \left(x^6\right)_0^1$$
$$= \pi$$

(c) the line $x = 1$

$$V = \int_{-1}^1 2\pi(1-x)(3x^4)dx$$
$$2\pi \int_{-1}^1 (1-x)(3x^4)dx$$
$$2\pi \left(\frac{3x^5}{5} - \frac{x^6}{6}\right)_{-1}^1$$
$$= 2\pi \left(\left(\frac{3}{5} - \frac{1}{2}\right) - \left(-\frac{3}{5} - \frac{1}{2}\right)\right) = \frac{12\pi}{5}$$

(d) the line $y = 3$

$$R(x) = 3 \qquad\qquad r(x) = 3 - 3x^4 = 3(1 - x^4)$$
$$V = \int_{-1} 1\pi \left(9 - 9(1-x^4)\right) dx$$
$$= 9\pi \int_{-1}^1 \left(1 - (1 - 2x^4 + x^8)\right) dx$$
$$= 9\pi \int_{-1}^1 \left(2x^4 - x^8\right) dx$$
$$= 9\pi \left(\frac{2x^5}{5} - \frac{x^9}{9}\right)_{-1}^1$$
$$= 18\pi \left(\frac{2}{5} - \frac{1}{9}\right) = \frac{2\pi \cdot 13}{5} = \frac{26\pi}{5}$$

7. Find the area between the curves $x = \frac{y^2}{4}$ and $y = x$

These curves are $2\sqrt{x} = y$ and $y = x$. They intersect where

$$2\sqrt{x} = x$$
$$4x = x^2$$
$$x^2 - 4x = 0$$
$$x(x-4) = 0$$

$$x = 0 \text{ and } x = 4$$

So the area between the curves is

$$\int_0^4 2\sqrt{x} - x dx$$

$$= \left(\frac{4}{3}x^{\frac{3}{2}} - \frac{x^2}{2}\right)_0^4$$

$$= \frac{4}{3} \cdot 8 - 4$$

$$= \frac{32}{3} - \frac{12}{3} = \frac{20}{3}$$

8. Find the lengths of the following curves on the given intervals;

   (a) $x = y^{\frac{2}{3}}, 1 \leq y \leq 8$

$$\frac{dx}{dy} = \frac{2}{3}y^{-\frac{1}{3}}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{4}{9}y^{-\frac{2}{3}}$$

$$L = \int_1^8 \sqrt{1 + \frac{4}{9y^{\frac{2}{3}}}} dy$$

$$= \int_1^8 \sqrt{\frac{9y^{\frac{2}{3}} + 4}{9y^{\frac{2}{3}}}} dy$$

$$= \int_1^8 \frac{\sqrt{9y^{\frac{2}{3}}}}{3y^{\frac{1}{3}}} dy$$

$$\frac{1}{3}\int_1^8 \sqrt{9y^{\frac{2}{3}} + 4}\left(y^{-\frac{1}{3}}\right) dy$$

$$u = 9y^{\frac{2}{3}} \qquad du = 6y^{-\frac{1}{3}}$$

$$= \frac{1}{18}\int_{13}^{40} u^{\frac{1}{2}} du$$

$$= \frac{1}{18}\left(\frac{2}{3}u^{\frac{3}{2}}\right)_{13}^{40}$$

$$= \frac{1}{27}\left(40^{\frac{3}{2}} - 13^{\frac{3}{2}}\right) \approx 7.634$$

   (b) $x = 5\cos t - \cos 5t, y = 5\sin t - \sin 5t, 0 \leq t \leq \frac{\pi}{2}$

$$\frac{dx}{dt} = -5\sin t + 5\sin 5t \qquad \frac{dy}{dt} = 5\cos t - 5\cos 5t$$

First let's just simplify the required square root:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(5\sin 5t - 5\sin t)^2 + (5\cos t - t\cos 5t)^2}$$

$$= 5\sqrt{\sin^2 5t - 2\sin t \sin 5t + \sin^2 t + \cos^2 t - 2\cos t\cos 5t + \cos^2 5t}$$

$$= 5\sqrt{2 - 2(\sin t\sin 5t + \cos t\cos 5t)}$$

$$= 5\sqrt{2(1-\cos 4t)}$$

$$= 5\sqrt{4\left(\frac{1}{2}\right)(1-\cos 4t)}$$

$$= 10\sqrt{\sin^2 2t}$$

$$= 10 \mid \sin 2t \mid = 10\sin 2t \text{ since } 0 \le t \le \frac{\pi}{2}$$

$$L = \int_0^{\frac{\pi}{2}} 10\sin 2t dt = (-5\cos 2t)_0^{\frac{\pi}{2}} = (-5)(-1) - (-5)(1) = 10$$

9. A force of 200 N will stretch a garage door spring 0.8 m beyond it unstressed length. How far will a 300 N force stretch the spring? How much work does it take to stretch the spring this far from its unstressed length?

$$F = kx \Rightarrow 200 = k(0.8) \Rightarrow k = 250N/m$$

$$300 = 250x \Rightarrow x = \frac{300}{250} = 1.2m$$

$$W = \int_0^{1.2} F(x)dx = \int_0^{1.2} 250x dx$$

$$= \left(125x^2\right)_0^{1.2} = 180J.$$

10. Find the limits of the following sequences:

(a) $a_n = \left(1 + \frac{(-1)^n}{n}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{(-1)^n}{n}\right)$$
$$= 1$$

(b) $a_n = \sin\frac{n\pi}{2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\frac{n\pi}{2}$$

This limit does not exist since the sequence is $\{1, -1, 1, -1, \ldots\}$

(c) $a_n = \dfrac{n + \ln n}{n}$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n + \ln n}{n} \\
&= \lim_{n\to\infty} 1 + \frac{\ln n}{n} \\
&= 1 + \lim_{n\to\infty} \frac{\ln n}{n} \\
&=^{LH} 1 + \lim_{n\to\infty} \frac{\frac{1}{n}}{1} \\
&= 1 + 0 = 1
\end{aligned}$$

(d) $a_n = \dfrac{\ln(2n^3)}{n}$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{\ln(2n^3)}{n} \\
&=^{LH} \lim_{n\to\infty} \frac{\frac{6n^2}{2n^3}}{1} \\
&= \lim_{n\to\infty} \frac{3}{n} = 0
\end{aligned}$$

(e) $a_n = \dfrac{(n+1)!}{n!}$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{(n+1)!}{n!} \\
&= \lim_{n\to\infty} (n+1) = \infty
\end{aligned}$$

Therefore the sequence diverges.

(f) $a_n = n\left(2^{\frac{1}{n}} - 1\right)$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} n\left(2^{\frac{1}{n}} - 1\right) \\
&= \lim_{n\to\infty} \frac{2^{\frac{1}{n}} - 1}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{2^{\frac{1}{n}}\left(\frac{-1}{n^2}\right)\ln 2}{-\frac{1}{n^2}} \\
&= \lim_{n\to\infty} 2^{\frac{1}{n}} \ln 2 = \ln 2
\end{aligned}$$

11. Determine if the following series converge or diverge. Provide reasons for your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-3)(2n-1)}$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-3)(2n-1)} = \sum_{n=1}^{\infty} \frac{A}{2n-3} + \frac{B}{2n-1}$$

$$A(2n-1) + B(2n-3) = 1$$

$$\text{If } x = \frac{1}{2} \Rightarrow -2B = 1 \Rightarrow B = \frac{-1}{2}$$

$$\text{If } x = \frac{3}{2} \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-3)(2n-1)} = \sum_{n=1}^{\infty} \frac{\frac{1}{2}}{2n-3} + \frac{\frac{-1}{2}}{2n-1}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2n-3} - \frac{1}{2} \cdot \frac{1}{2n-1}$$

Now $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-3} - \frac{1}{2n-1}$ converges by the integral test:

$$\int_1^{\infty} \frac{1}{2x-3} - \frac{1}{2x-1} = \lim_{t\to\infty} \left(\frac{1}{2} \ln \mid 2x-3 \mid - \frac{1}{2} \ln \mid 2x-1 \mid \right)_1^t$$

$$= \frac{1}{2} \lim_{t\to\infty} \ln \left| \frac{2x-3}{2x-1} \right| \Bigg|_1^t$$

$$= \frac{1}{2} \ln \left(\lim_{t\to\infty} \frac{2t-3}{2t-1} \right) - \frac{1}{2} \ln \frac{1}{1}$$

$$= \frac{1}{2} \ln \left(\lim_{t\to\infty} \frac{2}{2} \right) - 0$$

$$= \frac{1}{2} \ln 1 = 0$$

Therefore, the series also converges.

Notice that $f(x) = \dfrac{1}{2x-3} - \dfrac{1}{2x-1}$ is positive, decreasing and continuous.

(b) $\displaystyle\sum_{n=0}^{\infty} e^{-n}$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$$

Which is a geometric series with $r = \frac{1}{e} < 1$. Therefore, this series converges, and the sum is

$$\frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{-5}{n}$

$$= -5 \sum_{n=1}^{\infty} \frac{1}{n}$$

which is a multiple of the harmonic series, which diverges. Therefore this series also diverges.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3}$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^3}$$

which is a multiple of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ which converges by a $p-$series test. Therefore, this series also converges.

12. Determine if the following series converge absolutely, converge conditionally or diverge. Provide reasons for your answers.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n(\ln n)^2}$

Let's examine absolute convergence by looking at the integral test using $f(x) = \frac{1}{x(\ln x)^2}$, which is positive, decreasing and continuous.

Therefore the series converges or diverges with the integral:

$$\int_2^{\infty} \frac{x(\ln x)^2}{d} x$$

$$u = \ln x \qquad\qquad du = \frac{1}{x}$$

$$= \int_2^{\infty} \frac{1}{u^2} du$$

$$= \lim_{t\to\infty} -\frac{1}{u} |_2^t$$

$$= \lim_{t\to\infty} -\frac{1}{\ln t} + \frac{1}{\ln 2}$$

$$= \frac{1}{\ln 2}$$

Therefore the integral converges and the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n(\ln n)^2}$ converges absolutely

(b) $\displaystyle\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{n^3}$

Let's examine absolute convergence: $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

Use the comparison test:

$$\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$$

So

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^3} \leq \sum_{n=2}^{\infty} \frac{1}{n^2}$$

This converges by the $p-$series test and so $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ converges also. Therefore, $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{n^3}$ converges absolutely.

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n!}$

Let's examine absolute convergence and use the ratio test:

$$\lim_{n\to\infty} \frac{n+2}{(n+1)!} \cdot \frac{n!}{n+1}$$

$$= \lim_{n\to\infty} \frac{(n+2)}{(n+1)(n+1)}$$

$$= \lim_{n\to\infty} \frac{n+2}{n^2+2n+1}$$

$$=^{LH} \lim_{n\to\infty} \frac{1}{2n+2}$$

$$= 0 < 1$$

Therefore, it does converge absolutely.

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n (n^2+1)}{2n^2+n-1}$

Let's check absolute convergence with the ratio test:

$$\lim_{n\to\infty} \frac{(n+1)^2+1}{2(n+1)^2+(n+1)-1} \cdot \frac{2n^2+n-1}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+2}{2n^2+4n+2+n+1-1} \cdot \frac{2n^2+n-1}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+2}{2n^2+5n+2} \cdot \frac{2n^2+n-1}{n^2+1}$$

UGH!

Let's try Leibniz - let's look at condition 3.

$$\lim_{n\to\infty} \frac{n^2+1}{2n^2+n-1}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{2+\frac{1}{n}-\frac{1}{n^2}} = \frac{1}{2} \neq 0$$

Therefore the series diverges.

13. Determine the center and radius of convergence for the following power series. Specify where they converge absolutely and where they converge conditionally.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^{2n-2}}{(2n-1)!}$

Let's check absolute convergence with the ratio test:

$$\begin{aligned}
&\lim_{n\to\infty} \left| \frac{(x-1)^{2n}}{(2n+1)!} \cdot \frac{(2n-1)!}{(x-1)^{2n-2}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(x-1)^2}{(2n+1)2n} \right| \\
&= (x-1)^2 \lim_{n\to\infty} \frac{1}{(2n)(2n+1)} \\
&= (x-1)^2 \cdot 0 \\
&= 0 < 1
\end{aligned}$$

So it converges absolutely for all x.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-1)^n}{n^2}$

Check absolute converge by the ratio test:

$$\begin{aligned}
&\lim_{n\to\infty} \left| \frac{(3x-1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(3x-1)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(3x-1)n^2}{n^2+2n+1} \right| \\
&= |3x-1| \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} \\
&= |3x-1| \cdot 1 < 1
\end{aligned}$$

So this converges absolutely if

$$\begin{aligned}
-1 < 3x - 1 < 1 \\
0 < 3x < 2 \\
0 < x < \frac{2}{3}
\end{aligned}$$

Check for conditional convergence at the endpoints.

At $x = 0$, we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which converges by Leibniz.

at $x = \frac{2}{3}$ we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(3 \cdot \frac{2}{3} - 1)^n}{n^2} \\
&= \sum_{n=1}^{\infty} \frac{(2-1)^n}{n^2} \\
&= \sum_{n=1}^{\infty} \frac{(1)^n}{n^2}
\end{aligned}$$

which converges by $p-$series test.

So it converges absolutely from $0 < x < \frac{2}{3}$ and conditionally on the endpoints.

(c) $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

check absolute convergence using the $n-$th root test:

$$\lim_{n\to\infty} \left| \sqrt[n]{\frac{x^n}{n^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n} \right| = 0 < 1$$

So this series converges for all x.

14. Find Maclaurin series for the following functions: $\frac{1}{1-2x}$, $\cos\left(x^{\frac{5}{2}}\right)$, and $e^{\frac{\pi x}{2}}$

For $f(x) = \frac{1}{1-2x}$:This is the sum of a geometric series with $a_1 = 1$ and $r = 2x$

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n$$

where $\mid 2x \mid < a \rightarrow \mid x \mid < \frac{1}{2}$

For $f(x) = \cos\left(x^{\frac{5}{2}}\right)$ we have

$$\cos(x^{\frac{5}{2}}) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{\frac{5}{2}}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^n}{(2n)!}$$

For $e^{\frac{\pi x}{2}}$ we have:

$$e^{\frac{\pi x}{2}} = \sum_{n=0}^{\infty} \frac{\left(\frac{\pi x}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\pi^n}{2^n n!} x^n$$

15. Find the first four nonzero terms f the Taylor series generated by each of the following functions at x=2: $f(x) = \sqrt{3+x^2}$, $g(x) = \frac{1}{x+1}$, and $h(x) = \frac{1}{x}$

For $f(x) = \sqrt{3+x^2}$ (I'm changing this to near $x = -1$) 'cos I'm lazy:

$$f'(x) = x\left(3+x^2\right)^{\frac{-1}{2}}$$

$$f''(x) = -x^2\left(3+x^2\right)^{\frac{-3}{2}} + \left(3+x^2\right)^{-\frac{1}{2}}$$

$$f'''(x) = 3x^3\left(3+x^2\right)^{\frac{-5}{2}} - 3x\left(3+x^2\right)^{-\frac{3}{2}}$$

$$\sqrt{3+x^2} \approx 2 - \frac{x+1}{2} + \frac{3(x+1)^2}{2^3 \cdot 2!} + \frac{9(x+1)^3}{2^5 \cdot 3!}$$

For $g(x) = \frac{1}{x+1}$ near $x = 2$:

$$g'(x) = \frac{-1}{(x+1)^2}$$

$$g''(x) = \frac{2}{(x+1)^3}$$

$$g'''(x) = \frac{-6}{(x+1)^4}$$

$$\frac{1}{x+1} \approx \frac{1}{3} - \frac{1}{9} \cdot \frac{1}{2}(x-2) + \frac{2}{27} \cdot \frac{1}{3!}(x-2)^3 - \frac{2}{27} \cdot \frac{1}{4!}(x-2)^4$$

For $h(x) = \frac{1}{x}$:

$$h'(x) = -\frac{1}{x^2}$$

$$h''(x) = \frac{2}{x^3}$$

$$h'''(x) = \frac{-6}{x^4}$$

So near $x = 2$,

$$\frac{1}{x} \approx \frac{1}{2} - \frac{1}{4} \cdot \frac{1}{2}(x-2) + \frac{1}{4} \cdot \frac{1}{3!}(x-2)^3 - \frac{3}{8} \cdot \frac{1}{4!}(x-2)^4$$

You can simplify these, also!