Math 143, Fall, 2006 **Review of Integrals** Name: **Solutions**

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1. Find the most general antiderivative for the following functions:

   (a) $f(x) = 5x^{\frac{1}{4}} - 7x^{\frac{3}{4}}$

$$\frac{4}{5}5x^{\frac{5}{4}} - 7 \cdot \frac{4}{7}x^{\frac{7}{4}} + C$$
$$= 4x^{\frac{5}{4}} - 4x^{\frac{7}{4}} + C$$

   (b) $f(t) = 7\cos t - 5\sin t$

$$7\sin t + 5\cos t + C$$

   (c) $f(\theta) = e^{\theta} + \sec\theta\tan\theta$

$$e^{\theta}\sec\theta + C$$

   (d) $f(x) = \dfrac{x^2 + x + 1}{x}$

$$f(x) = \frac{x^2 + x + 1}{x} = x + 1 + \frac{1}{x}$$

   so its anti-derivative is

$$\frac{1}{2}x^2 + x + \ln \mid x \mid + C$$

2. Find $f(x)$:

   (a) $f'(x) = \dfrac{4}{\sqrt{1 - x^2}}$ when $f(\frac{1}{2}) = 1$

$$f(x) = -\frac{1}{x} + \ln \mid x \mid + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$$

   (b) $f''(x) = x^2 + 3\cos x$ when $f(1) = 1$ and $f'(1) = 5$

$$f'(x) = \frac{1}{3}x^3 + 3\sin x + C$$

   Since $f'(1) = 5$

$$f'(1) = \frac{1}{3}1^3 + 3\sin 1 + C = 5$$
$$C = 5 - \frac{1}{3} - 3\sin 1 = \frac{14}{3} - 3\sin 1$$

   Therefore

$$f'(x) = \frac{1}{3}x^3 + 3\sin x + \frac{14}{3} - 3\sin 1$$

$$f(x) = \frac{1}{12}x^4 - 3\cos x + \frac{14}{3}x - (3\sin 1)x + C$$

$f(1) = 1$, so

$$f(1) = \frac{1}{2} - 3\cos 1 + \frac{14}{3} - 3\sin 1 + C = 1$$
$$C = 1 - \frac{1}{2} - \frac{14}{3} + 3\cos 1 + 3\sin 1 = -\frac{25}{6} + 3\cos 1 + 3\sin 1$$
$$f(x) = \frac{1}{12}x^4 - 3\cos x + \frac{14}{3}x - (3\sin 1)x - \frac{25}{6} + 3\cos 1 + 3\sin 1$$

3. Find all possible functions with the given derivatives:

(a) $f'(x) = \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2$

$$f(x) = -\frac{1}{x} + \ln \mid x \mid + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$$

(b) $f'(x) = \sec x \tan x$

$$f(x) = \sec x + C$$

(c) $f'(x) = 5^x + xe^{-x^2}$

$$f(x) = \frac{1}{\ln 5}5^x - \frac{1}{2}e^{-x^2} + C$$

(d) $f'(x) = \sin x$

$$f(x) = -\cos x + C$$

(e) $f'(x) = \sqrt{2x+3} + \frac{1}{\sqrt{2x+3}}$

$$f(x) = \frac{1}{2} \cdot \frac{2}{3}(2x+3)^{\frac{3}{2}} + 2\sqrt{2x+3} + C$$

$$= \frac{1}{3}(2x+3)^{\frac{3}{2}} + 2\sqrt{2x+3} + C$$

4. Evaluate the following integrals. Use differentiation to check your answers:

(a) $\int xe^{-x^2} dx$

$$= -\frac{1}{2}e^{-x^2} + C$$

(b) $\int \cos x dx$

$$= \sin x + C$$

(c) $\int \frac{3}{(2-x)^2} dx$

$$= \frac{3}{2-x} + C$$

(d) $\int \tan x dx$

$$= \int \frac{\sin x}{\cos x}\, dx$$

$$= -\ln \mid \cos x \mid + C$$

(e) $\int \frac{1}{\sqrt{1-3x}} dx$

$$= -\frac{2}{3}\sqrt{1-3x} + C$$

(f) $\int \frac{1}{\sqrt{1-9x^2}}dx$

$$= \int \frac{1}{\sqrt{1-(3x)^2}}\ dx$$
$$= \frac{1}{3}\sin^{-1}(3x) + C$$

(g) $\int (e^{-x} + 4^x)dx$

$$= -e^{-x} + \frac{1}{\ln 4}4^x + C$$

(h) $\int (\cos^2 x + \sin^2 x)dx$

$$= \int 1\ dx$$
$$= x + C$$

(i) $\int \pi dx$

$$= \pi x + C$$

(j) $\int \frac{4}{x + x\ln^2 x}dx$

$$= \int \frac{4}{x(1+\ln^2 x)}dx$$
$$= \int 4 \cdot \frac{1}{x} \cdot \frac{1}{1+(\ln x)^2}\ dx$$
$$= 4\tan^{-1}(\ln x) + C$$

(k) $\int \frac{x^2+4}{x}dx$

$$= \int x + \frac{4}{x}\ dx$$
$$= \frac{1}{2}x^2 + 4\ln \mid x \mid + C$$

5. Find the average value of $f(x)$ over the given interval where $f(x) = -x^2 + 10x + 11$ on the interval $[0, 10]$

$$\begin{aligned}
f_{\text{avg}} &= \frac{1}{10-0}\int_0^{10} -x^2 + 10x + 11 \\
&= \frac{1}{10}\left(-\frac{1}{3}x^3 + 5x^2 + 11x\right|_0^{10} \\
&= \frac{1}{10}\left(-\frac{1}{3}10^3 + 5(100) + 11(10) - 0\right) \\
&= \frac{1}{10}\cdot\frac{830}{3} \\
&= \frac{83}{3}
\end{aligned}$$

6. Let $f(t) = \frac{1}{2}\sin^2 t$. Then $F'(t) = \sin t \cos t$. Use the Fundamental Theorem of Calculus to find $\int_{\frac{\pi}{2}}^{\pi} \sin t \cos t \; dt$.

$$\begin{aligned}
\int_{\frac{\pi}{2}}^{\pi} \sin t \cos t \; dt &= \left(\frac{1}{2}\sin^2 t\right|_{\frac{\pi}{2}}^{\pi} \\
&= \frac{1}{2}(\sin \pi)^2 - \frac{1}{2}\left(\sin \frac{\pi}{2}\right)^2 \\
&= \frac{1}{2}0^2 - \frac{1}{2}1^2 \\
&= -\frac{1}{2}
\end{aligned}$$

7. If the average value of $f(x)$ over $[-3, 5]$ is 4, what is $\int_{-3}^{5} f(x)dx$?

$$\begin{aligned}
\frac{1}{5-(-3)}\int_{-3}^{4} f(x) &= 4 \\
\frac{1}{8}\int_{-3}^{4} f(x) &= 4 \\
\int_{-3}^{4} f(x) &= 4 \cdot 8 = 32
\end{aligned}$$

8. Compute the following indefinite integrals:

(a) $\int \frac{\csc\theta}{\csc\theta - \sin\theta} d\theta$

$$\begin{aligned}
&= \int \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \sin\theta} \; d\theta \\
&= \int \frac{\frac{1}{\sin\theta}}{\frac{1-\sin^2\theta}{\sin\theta}} \; d\theta \\
&= \int \frac{1}{\sin\theta} \cdot \frac{\sin\theta}{1-\sin^2\theta} d\theta \\
&= \int \frac{1}{\cos^2\theta} \; d\theta \\
&= \int \sec^2\theta \; d\theta \\
&= \tan\theta + C
\end{aligned}$$

(b) $\int \frac{e^x}{1+e^{2x}} dx$

$$\begin{aligned}
&= \int \frac{e^x}{1+(e^x)^2} dx \\
&= \tan^{-1} e^x + C
\end{aligned}$$

(c) $\int x^{-2} + x^3 + 2x + 5 \ dx$

$$= -x^{-1} + \frac{1}{4}x^4 + x^2 + 5x + C$$

(d) $\int e^x + x^e + edx$

$$= e^x + \frac{1}{e+1}x^{e+1} + ex + C$$

(e) $\int \frac{1}{x}dx$

$$= \ln \mid x \mid + C$$

(f) $\int 3\sin x - 5\cos x \ dx$

$$= -3\cos x - 5\sin x + C$$

(g) $\int x(x^2-3)^{49} \ dx$

$$= \frac{1}{2} \cdot \frac{1}{50}(x^2-3)^{50} + C$$

$$= \frac{1}{100}\left(x^2-3\right)^{50} + C$$

(h) $\int \frac{x-2}{x^2-4x}\ dx$

$$= \int \frac{1}{2} \cdot \frac{2(x-2)}{x^2-4x}\ dx$$
$$= \int \frac{1}{2} \cdot \frac{2x-4}{x^2-4x}\ dx$$
$$= \frac{1}{2} \ln \mid x^2 - 4x \mid + C$$

(i) $\int x \sin(x^2)\ dx$

$$= -\frac{1}{2}\cos(x^2) + C$$

9. Compute the following definite integrals:

(a) $\int_{-4}^{6} \mid x+2 \mid dx$
Using the graph and the fact that the integral is the area under the curve, we have that this integral is equal to the area under the two triangles:

$$\frac{1}{2} \cdot 2 \cdot 2 + \frac{1}{2} \cdot 8 \cdot 8$$
$$= 2 + 32$$
$$= 34$$

(b) $\int_{0}^{\frac{\pi}{2}} e^{\sin x} \cos x\ dx$

$$= e^{\sin x}\Big|_0^{\frac{\pi}{2}}$$
$$= e^{\sin \frac{\pi}{2}} - e^{\sin 0}$$
$$= e^1 - e^0$$
$$e - 1$$

(c) $\int_{0}^{10} -x^2 + 10x + 11\ dx$

$$= \left(-\frac{1}{3}x^3 + 5x^2 + 11x\right|_0^{10}$$
$$= -\frac{1}{3}10^3 + 500 + 110 - 0$$
$$= \frac{830}{3}$$

(d) $\int_{1}^{2} \sqrt{x}\ dx$

$$= \left(\frac{2}{3}x^{\frac{3}{2}}\right|_1^2$$
$$= \frac{2}{3}2^{\frac{3}{2}} - \frac{2}{3}1^{\frac{3}{2}}$$
$$= \frac{2}{3}\sqrt{8} - \frac{2}{3}$$

(e) $\int_2^6 3t^2 + 4t\ dt$

$$= t^3 + 2t^2\Big|_2^6$$

$$= 6^3 + 2 \cdot 6^2 - 2^3 - 2 \cdot 2^2$$

$$= 272$$

(f) $\int_0^3 x(x^2-3)^{49}\ dx$

$$= \frac{1}{100}\left(x^2-3\right)^{50}\Big|_0^3$$

$$= \frac{1}{100}\left(3^2-3\right)^{50} - \frac{1}{100}\left(-3\right)^{50}$$

(g) $\int_1^4 \frac{x-2}{x^2-4x}\ dx$

$$= \int_1^4 \frac{1}{2} \cdot \frac{2(x-2)}{x^2-4x}\ dx$$

$$= \frac{1}{2}\ln \mid x^2 - 4x \mid \Big|_1^4$$

$$= \frac{1}{2}\ln \mid 1^2 - 4 \cdot 1 \mid - \frac{1}{2}\ln \mid 4^2 - 4 \mid$$

$$= \frac{1}{2}\ln 3 - \frac{1}{2}\ln 12$$

$$= \ln\sqrt{3} - \ln\sqrt{12}$$

$$= \ln\sqrt{\frac{3}{12}}$$

$$= \ln\sqrt{\frac{1}{4}}$$

$$= \ln\frac{1}{2}$$

(h) $\int_{\frac{\pi}{2}}^{\pi} x\sin(x^2)\ dx$

$$= -\frac{1}{2}\cos(x^2)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{1}{2}\cos(\pi^2) + \frac{1}{2}\cos\left(\frac{\pi}{2}\right)^2$$

10. Suppose that f and g are continuous functions and that $\int_0^2 f(x)dx = \sqrt{2}$, $\int_0^5 f(x)dx = \sqrt{5}$ and $\int_0^2 g(x)dx = 1$. Find the following:

(a) $\int_0^2 (7f(x) - 11g(x))dx$

$$= 7\int_0^2 f(x)dx - 11\int_0^2 g(x)dx$$
$$= 7 \cdot \sqrt{2} - 11 \cdot 1$$
$$= 7\sqrt{2} - 11$$

(b) $\int_2^5 f(x)dx$

$$\int_0^2 f(x)dx + \int_2^5 f(x)dx = \int_0^5 f(x)dx$$
$$\sqrt{2} + \int_2^5 f(x)dx = \sqrt{5}$$
$$\int_2^5 f(x)dx = \sqrt{5} - \sqrt{2}$$

11. Compute the following derivatives:

(a) $\frac{d}{dx}\int_1^x (t^2 - 1)^{19}dt$

$$= (x^2 - 1)^{19}$$

(b) $\frac{d}{dx}\int_0^x \sin\theta^2 d\theta$

$$= \sin x^2$$

(c) $\frac{d}{dx}\int_x^\pi \frac{1}{1+t^4}dt$

$$= \frac{d}{dx} - \int_\pi^x \frac{1}{1+t^4}dt$$
$$= -\frac{1}{1+x^4}$$

(d) $\frac{d}{dx}\int_1^{\sqrt{x}} \frac{r^2}{r^2+1}dr$

$$= \frac{(\sqrt{x})^2}{(\sqrt{x})^2+1} \cdot \frac{1}{2\sqrt{x}}$$
$$= \frac{x}{x+1} \cdot \frac{1}{2\sqrt{x}}$$

(e) $\frac{d}{dx}\int_{-5}^{\sin x} u\cos u^3 du$

$$= \sin x \cos(\sin^3 x) \cdot \cos x$$

(f) $\frac{d}{dx}\int_{2}^{129} \sin x dx$

$$= 0 \text{ since it is a constant}$$

12. Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the $x-$axis and the line $y = x - 2$.

    We need to do this in two pieces:

$$\int_0^2 \sqrt{x}\ dx + \int_2^4 \sqrt{x} - (x-2)\ dx$$

$$= \left(\frac{2}{3}x^{\frac{3}{2}}\right|_0^2 + \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x\right|_2^4$$

$$= \frac{2}{3}2^{\frac{3}{2}} - 0 + \left(\frac{2}{3}4^{\frac{3}{2}} - \frac{1}{2}4^2 + 2\cdot 4 - \frac{2}{3}2^{\frac{3}{2}} - \frac{1}{2}2^2 + 2\cdot 2\right)$$

$$= \frac{2}{3}\cdot 2^{\frac{3}{2}} + \frac{16}{3} - 8 + 8 - \frac{2}{3}\cdot 2^{\frac{3}{2}} - 2 + 4$$

$$= \frac{16}{3} + 2$$

$$= \frac{22}{3}$$

13. Find the area of the region bounded by the curves $y = x^3$ and $x = y^3$.

$$\int_0^1 \sqrt[3]{x} - x^3\ dx$$

$$= \frac{3}{4}x^{\frac{4}{3}} - \frac{1}{4}x^4\Big|_0^1$$

$$= \frac{3}{4} - \frac{1}{4} - 0$$

$$= \frac{1}{2}$$