1. Be able to state the Extreme Value Theorem.

   Let f be continuous on $[a, b]$. Then there exists $c \in [a, b]$ and $d \in [a, b]$ so that f attains a maximum at c and a minimum at d, i.e. $f(c) \geq f(x) \forall x \in [a, b]$ and $f(d) \leq f(x) \forall x \in [a, b]$.

2. Be able to state Rolle's theorem.

   Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is a $c \in (a, b)$ so that $f'(c) = 0$.

3. Be able to state the Mean Value Theorem.

   Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is a $c \in (a, b)$ so that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

4. Be able to do word problems like the worksheet for section 2.7.

5. Be able to do word problems like the worksheet for section 3.5.

6. Be able to take derivatives of all of the functions on the pink "review of derivatives" sheet.

7. Find the values of θ, $\sin\theta$, and $\cos\theta$ given that:

   (a) $\theta = \sin^{-1}\frac{1}{2}$

$$\theta = \frac{\pi}{6}$$

$$\sin\theta = \frac{1}{2}$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

   (b) $\theta = \sec^{-1}\sqrt{2}$

$$\theta = \sec^{-1}\sqrt{2}$$

$$\sec\theta = \sqrt{2}$$

$$\frac{1}{\cos\theta} = \sqrt{2}$$

$$\cos\theta = \frac{\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4}$$

$$\sin\theta = \frac{\sqrt{2}}{2}$$

(c) $\theta = \cot^{-1} \frac{1}{\sqrt{3}}$

$$\theta = \cot^{-1} \frac{1}{\sqrt{3}}$$
$$\cot\theta = \frac{1}{\sqrt{3}}$$
$$\frac{x}{y} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\sqrt{3}}$$
$$\theta = \frac{\pi}{3}$$
$$\sin\theta = \frac{\sqrt{3}}{2}$$
$$\cos\theta = \frac{1}{2}$$

8. Find $\frac{dy}{dx}$ for each of the following;

(a) $xy + 2x + 3y = 1$

$$y(x+3) = 1 - 2x$$
$$y = \frac{1-2x}{x+3}$$
$$\frac{dy}{dx} = \frac{-2(x+3) - 1(1-2x)}{(x+3)^2}$$
$$= \frac{-2x - 6 - 1 + 2x}{(x+3)^2}$$
$$= \frac{-7}{(x+3)^2}$$

(b) $\sqrt{xy} + y^2 = 1$

$$\frac{d}{dx}\left(\sqrt{xy} + y^2\right) = \frac{d}{dx}(1)$$
$$\frac{1}{2}(xy)^{-\frac{1}{2}}\left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$
$$\frac{y}{2\sqrt{xy}} + \frac{x}{2\sqrt{xy}}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}\left(\frac{x}{2\sqrt{xy}} + 2y\right) = -\frac{y}{2\sqrt{xy}}$$
$$\frac{dy}{dx}\left(\frac{x + 4y\sqrt{xy}}{2\sqrt{xy}}\right) = -\frac{y}{2\sqrt{xy}}$$
$$\frac{dy}{dx} = -\frac{y}{2\sqrt{xy}} \cdot \frac{2\sqrt{xy}}{x + 4y\sqrt{xy}}$$
$$\frac{dy}{dx} = -\frac{y}{x + 4y\sqrt{xy}}$$

(c) $\sin^2 x + \cos^2 y = 1$

$$2\sin x\cos x + 2\cos y(-\sin y)\frac{dy}{dx} = 0$$
$$2\sin x\cos x - 2\cos y\sin y\frac{dy}{dx} = 0 =$$
$$2\sin x\cos x = 2\cos y\sin y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{\sin x\cos x}{\cos y\sin y}$$

(d) $\dfrac{x^3 - xy}{x} = 3y$

$$x^2 - y = 3y$$
$$x^2 = 4y$$
$$\frac{1}{4}x^2 = y$$
$$\frac{dy}{dx} = \frac{1}{4}2x$$
$$\frac{dy}{dx} = \frac{1}{2}x$$

(e) $x(t) = 2t^7 - 4t^2 + 5$ $\qquad y(t) = 3t^2 + 5$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$\frac{dy}{dt} = 6t$$
$$\frac{dx}{dt} = 14t^6 - 8t$$
$$\frac{dy}{dx} = \frac{6t}{14t^6 - 8t}$$
$$\frac{dy}{dx} = \frac{6}{14t^5 - 8}$$

(f) $x(t) = \cos(5t^2 + t)$ $\qquad y(t) = \cos^2(7t^3 - t^2)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$\frac{dy}{dt} = 2\cos(7t^3 - t^2) \cdot -\sin(7t^3 - t^2) \cdot (21t^2 - 2t)$$
$$= -2(21t^2 - 2t)\cos(7t^3 - t^2)\sin(7t^3 - t^2)$$
$$\frac{dx}{dt} = -\sin(5t^2 + t) \cdot (10t + 1)$$
$$= -(10t + 1)\sin(5t^2 + t)$$
$$\frac{dy}{dx} = \frac{-2(21t^2 - 2t)\cos(7t^3 - t^2)\sin(7t^3 - t^2)}{-(10t + 1)\sin(5t^2 + t)}$$
$$= \frac{(41t^2 - 4t)\cos(7t^3 - t^2)\sin(7t^3 - t^2)}{(10t + 1)\sin(5t^2 + 1)}$$

(g) $y = x^3 - 3x^2 - 1$

$$\frac{dy}{dx} = 3x^2 - 6x$$

9. Find equations of the tangent line to the curve at $x = a$.

(a) $f(x) = -\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{x^2}\right)$ for $a = 2$.

$$\begin{aligned} f(2) &= -\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{4}\right) \\ &= -\left(\frac{3}{2}\right)\left(\frac{5}{4}\right) \\ &= -\frac{15}{8} \end{aligned}$$

$$\begin{aligned} f'(x) &= -\left(-\frac{1}{x^2}\right)\left(1 + \frac{1}{x^2}\right) - \left(-\frac{2}{x^3}\right)\left(1 + \frac{1}{x}\right) \\ &= \frac{1}{x^2}\left(1 + \frac{1}{x^2}\right) + \frac{2}{x^3}\left(1 + \frac{1}{x}\right) \\ &= \frac{1}{x^2} + \frac{1}{x^4} + \frac{2}{x^3} + \frac{2}{x^4} \\ &= \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} \\ &= \frac{x^2 + 2x + 3}{x^4} \\ f'(x) &= \frac{4 + 4 + 3}{16} = \frac{11}{16} \end{aligned}$$

$$y + \frac{15}{8} = \frac{11}{16}(x - 2)$$

(b) $g(x) = \cos \pi x + 3$ for $a = 2$.

$$g(2) = \cos(2\pi) + 3 = 1 + 3 = 4$$

$$\begin{aligned} g'(x) &= -\pi \sin(\pi x) \\ g'(2) &= -\pi \sin(2\pi) = 0 \end{aligned}$$

$$\begin{aligned} y - 4 &= 0 \\ y &= 4 \end{aligned}$$

10. Find the points where the tangent to the curve has the following properties:

(a) $f(x) = -x^2 - 6$ is parallel to the line $y = 4x - 1$.
We want to find where the slope is 4.:

$$f'(x) = -2x$$

$$\begin{aligned} -2x &= 4 \\ x &= -2 \end{aligned}$$

(b) $g(x) = x^3 - 3x$ is perpendicular to the line $5y - 3x - 8 = 0$.
This line has slope found by putting the equation in slope-intercept form:

$$\begin{aligned} 5y &= 3x + 8 \\ y &= \frac{3}{5}x + \frac{8}{5} \\ m &= \frac{3}{5} \end{aligned}$$

So the perpendicular slope is:

$$m = -\frac{5}{3}$$

So we want to find where the slope of $g(x)$ is $-\frac{5}{3}$:

$$\begin{aligned} g'(x) &= 3x^2 - 3 \\ 3x^2 - 3 &= -\frac{5}{3} \\ 3x^2 &= -\frac{5}{3} + 3 \\ 3x^2 &= \frac{4}{3} \\ x^2 &= \frac{4}{9} \\ x &= \pm\frac{2}{3} \end{aligned}$$

(c) $g(x) = \frac{5x}{x^2+1}$ is horizontal.
We want the slope of the tangent line to be 0.

$$\begin{aligned} g'(x) &= \frac{5(x^2+1) - (2x)(5x)}{(x^2+1)^2} \\ &= \frac{5x^2 + 5 - 10x^2}{(x^2+1)^2} \\ &= \frac{5 - 5x^2}{(x^2+1)^2} \end{aligned}$$

$$\frac{5 - 5x^2}{(x^2+1)^2} = 0$$

$$5 - 5x^2 = 0$$
$$5 = 5x^2$$
$$1 = x^2$$
$$x = \pm 1$$

11. Does $f(x) = \dfrac{x+4}{x+1}$ ever have a horizontal tangent line? Justify your answer.

We would need the derivative to be 0:

$$\begin{aligned} f'(x) &= \frac{1(x+1) - 1(x+4)}{(x+1)^2} \\ &= \frac{x+1-x-4}{(x+1)^2} \\ &= \frac{-3}{(x+1)^2} \end{aligned}$$

This is never 0 so this function never has horizontal tangent lines.

12. (a) Find the intervals where $f(x) = \pi^{\cos x}$ is increasing and where it is decreasing.

$$f'(x) = \pi^{\cos x} \ln \pi(-\sin x)$$

$f'(x) = 0$ exactly when $\sin x = 0$. This occurs at πk for any integer k. Therefore this changes sign every π units. It is positive when $-\sin x$ is positive, that is on the following intervals:

$$((2k-1)\pi, 2k\pi)$$

(b) Find the intervals where $f(x) = 2x^3 + 3x^2 - 37x + 24$ has slope less than or equal to -1.

$$f'(x) = 6x^2 + 6x - 37$$

We want this to be less than or equal to -1.

$$6x^2 + 6x + 37 \leq -1$$
$$6x^2 + 6x + 38 \leq 0$$
$$3x^2 + 3x + 19 \leq 0$$

This is never negative (it has no $x-$intercepts, and is always positive). Therefore, this function never has slope less than or equal to -1.

13. Find all values of x where $f(x) = \frac{-5x}{5x-15}$ is concave up.

$$\begin{aligned} f'(x) &= \frac{-5(5x-15) - 5(-5x)}{(5x-15)^2} \\ &= \frac{-25x + 75 + 25x}{(5x-15)^2} \\ &= \frac{75}{(5x-15)^2} \end{aligned}$$

$$= 75(5x - 15)^{-2}$$

$$f''(x) = -150(5x - 15)(5)$$
$$= -750(5x - 15)$$

The function is concave up if $f''(x) > 0$:

$$-750(5x - 15) > 0$$
$$5x - 15 < 0$$
$$5x < 15$$
$$x < 3$$

Therefore the function is concave up for $x < 3$.

14. If $f(x)$ is increasing for all x, find where $f \circ f$ is increasing.

    Since f is increasing, $f'(x) > 0$ for all x. We want to say something about the derivative of $f \circ f$:

    $(f \circ f)' = f'(f(x)) \cdot f'(x)$

    Since $f'(x) > 0$ for any input, $f'(f(x)) > 0$ and $f'(x) > 0$. Therefore the product is also positive. Therefore $(f \circ f)' > 0$ for all x, and this function is increasing for all x.

15. A farmer wants to build a rectangular garden which is to have a 4 foot opening on one side. If the fence costs \$3 per foot for the side with the opening and \$2 per foot for the other 3 sides, find the dimensions of the largest garden he can build if he only has \$300 to spend.

    Let x be the length of the side without the opening, and y be the length of the side with the opening. Then the farmer needs $y - 4$ feet of fencing on the side with the opening, and $2x + y$ feet of fencing for the rest. Therefore,

$$300 = (2x + y)2 + (y - 4)3$$
$$300 = 4x + 2y + 3y - 12$$
$$312 = 4x + 5y$$
$$312 - 5y = 4x$$
$$\frac{312}{4} - \frac{5}{4}y = x$$

and the area is $A = xy$

$$A = xy$$
$$A = y\left(\frac{312}{4} - \frac{5}{4}y\right)$$
$$= \frac{312}{4}y - \frac{5}{4}y^2$$
$$A' = \frac{312}{4} - \frac{5}{2}y$$

$$A' = 0$$
$$\frac{312}{4} - \frac{5}{2}y = 0$$
$$\frac{312}{4} = \frac{5}{2}y$$
$$\frac{632}{20} = y = \frac{312}{10} = \frac{156}{5} = 31.2$$

The other dimension is

$$x = \frac{312}{4} - \frac{5}{4} \cdot \frac{156}{5} = \frac{312}{4} - \frac{156}{4}$$
$$x = \frac{156}{4} = 39$$

Therefore the dimensions are 31.2 feet by 39 feet.

16. A closed rectangular box is to be constructed with a base that is twice as long as it is wide. If the total surface area is to be 24 square centimeters, find the dimensions of the box that will maximize the volume.

Let w be the length of one side of the bottom of the box. Then the length of the other side is $2w$, and we have:

$$V = 2w \cdot w \cdot h$$

and

$$SA = 2 \cdot (w \cdot 2w) + 2(w \cdot h) + 2(2w \cdot h) = 24$$
$$4w^2 + 2wh + 4wh = 24$$
$$4w^2 + 6wh = 24$$
$$6wh = 24 - 4w^2$$
$$h = \frac{4}{w} - \frac{2}{3}w$$

Therefore

$$V = 2w^2h = 2w^2\left(\frac{4}{w} - \frac{2}{3}w\right)$$
$$V = 8w - \frac{4}{3}w^3$$

This is the function that we wish to maximize. Therefore:

$$V' = 8 - 4w^2$$

$$V' = 0 \Longrightarrow 8 - 4w^2 = 0$$
$$8 = 4w^2$$
$$2 = w^2$$
$$w = \pm\sqrt{2}$$

Clearly we cannot use the negative solution, so we have $w = \sqrt{2}$. We need to ensure that this is a maximum, and so

$$V'' = -8w$$
$$V''(\sqrt{2}) = -8\sqrt{2} < 0$$

So the function is concave down there and our critical point is a maximum.

Therefore the dimensions to maximize the volume are

$$\sqrt{2} \times 2\sqrt{2} \times \left(\frac{4}{\sqrt{2}} - \frac{2\sqrt{2}}{3} \right)$$

17. Use the definition of the derivative to prove that if a function is periodic, then its derivative is periodic with the same period.

A function being periodic means that $f(x+c) = f(x)$ where c is the period of the function.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We want to see that $f'(x+c) = f'(x)$ where c is also the period of the function, so let's examine this function:

$$f'(x+c) = \lim_{h \to 0} \frac{f(x+c+h) - f(x+c)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f'(x)$$

since $f(x+c+h) = f(x+h)$ since the function is periodic. Therefore if a function is periodic, then the derivative is also periodic and has the same period.

18. Find the most general antiderivative for the following functions:

(a) $f(x) = 5x^{\frac{1}{4}} - 7x^{\frac{3}{4}}$

$$4x^{\frac{5}{4}} - 4x^{\frac{7}{4}} + C$$

(b) $f(t) = 7\cos t - 5\sin t$

$$-7\sin t - 5\cos t + C$$

(c) $f(\theta) = e^{\theta} + \sec\theta\tan\theta$

$$e^{\theta} + \sec\theta + C$$

(d) $f(x) = \frac{x^2+x+1}{x}$

$$f(x) = x + 1 + \frac{1}{x}$$

It's antiderivative is:

$$\frac{1}{2}x^2 + x + \ln \mid x \mid + C$$

19. Find $f(x)$:

(a) $f'(x) = \frac{4}{\sqrt{1-x^2}}$ when $f(\frac{1}{2}) = 1$

$$f(x) = 4\sin^{-1} x + C$$
$$1 = 4\sin^{-1}\frac{1}{2} + C$$
$$1 = 4\left(\frac{\pi}{6}\right) + C$$
$$1 = \frac{2\pi}{3} + C$$
$$1 - \frac{2\pi}{3} = C$$
$$f(x) = 4\sin^{-1} x + 1 - \frac{2\pi}{3}$$

(b) $f''(x) = x^2 + 3\cos x$ when $f(1) = 1$ and $f'(1) = 5$

$$f'(x) = \frac{1}{3}x^3 - 3\sin x + C$$
$$5 = \frac{1}{3}(1^3) - 3\sin 1 + C$$
$$\frac{14}{3} + 3\sin 1 = C$$

$$f'(x) = \frac{1}{3}x^3 - 3\sin x + \frac{14}{3} + 3\sin 1$$
$$f(x) = \frac{1}{12}x^4 - 3\cos x + \frac{14}{3}x + (3\sin 1)x + C$$

$$1 = \frac{1}{12} - 3\cos 1 + \frac{14}{3} + 3\sin 1 + C$$
$$1 - \frac{1}{12} + 3\cos 1 - 3\sin 1 - \frac{14}{3} = C$$

$$f(x) = \frac{1}{12}x^4 - 3\cos x + \frac{14}{3}x + (3\sin 1)x + 1 - \frac{1}{12} + 3\cos 1 - 3\sin 1 - \frac{14}{3}$$

20. Differentiate $f(x) = \dfrac{x}{1+x}$.

$$f'(x) = \frac{1(1+x) - 1(x)}{(1+x)^2}$$
$$= \frac{1+x-x}{(1+x)^2}$$
$$= \frac{1}{(1+x)^2}$$

21. Differentiate $g(x) = \pi x^e - e^\pi x$.

$$g'(x) = \pi e x^{e-1} - e^\pi$$

22. Differentiate $h(x) = x \cdot \sin x$.

$$h'(x) = \sin x + x \cos x$$

23. Differentiate $l(x) = \dfrac{\sin x}{1 + \cos x}$.

$$\begin{aligned} l'(x) &= \frac{\cos x(1+\cos x) + \sin x(\sin x)}{(1+\cos x)^2} \\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} \\ &= \frac{\cos x + 1}{(1+\cos x)^2} \\ &= \frac{1}{1+\cos x} \end{aligned}$$

24. What can be said about the extreme values of a function that is continuous on a closed interval?

The function attains its absolute maximum and its absolute minimum (Extreme value theorem)

25. What is a differential equation? What is a solution to a differential equation?

A differential equation is an equation involving a derivative. A solution to a differential equation is a function whose derivative is given by the differential equation.

26. If you set the second derivative equal to zero, solve the equation, and get the solution $x = c$, must $x = c$ be an inflection point? If yes, prove it. If no, give an example of a function $f(x)$ for which there is a c with $f''(c) = 0$ but c is not an inflection point.

No - a point of inflection only occurs when the second derivative changes sign, i.e. the function changes from concave up to concave down or vice versa. There are many possible examples - try $f(x) = x^4$ at $x = 0$.

27. Can different functions have the same derivative? If yes, give an example of two different functions with the same derivative. If no, prove that different functions cannot have the same derivative.

Yes. For example:

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= x^2 + 13 \\ f'(x) &= g'(x) = 2x \end{aligned}$$

28. If $y = f(x)$ is a differentiable function, is it possible that there exists a point c such that $f'(c) < 0$ and $f''(c) > 0$? If yes, give an example. If no, explain why such a point cannot exist.

Yes, this means that the function is decreasing by concave up somewhere. Look at $f(x) = -x^3$.

$$\begin{aligned} f'(x) &= -3x^2 < 0 \\ f''(x) &= -6x \end{aligned}$$

For $x = -1$ the function has negative derivative and positive second derivative.

29. Find all possible functions with the given derivatives:

(a) $f'(x) = \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2$

$$f(x) = -\frac{1}{x} + \ln \mid x \mid + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$$

(b) $f'(x) = \sec x \tan x$

$$f(x) = \sec x + C$$

(c) $f'(x) = 5^x + xe^{-x^2}$

$$f(x) = \frac{1}{\ln 5}5^x - \frac{1}{2}e^{-x^2}$$

(d) $f'(x) = \sin x$

$$f(x) = -\cos x + C$$

(e) $f'(x) = \sqrt{2x+3} + \frac{1}{\sqrt{2x+3}}$

$$f(x) = \frac{1}{3}(2x+3)^{\frac{3}{2}} + \sqrt{2x+3} + C$$