Math 142
Spring, 2005 Name: **Solutions**
Optimization Problems

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1. Find the maximum slope of the graph $f(x) = 24x^2 - x^4$ for $x \geq 0$.

$$m = f'(x) = 48x - 4x^3$$
$$m' = f''(x) = 48 - 12x^2 = 12(4 - x^2)$$
$$m' = 0 \text{ if } x = \pm 2$$

but $x = -2$ is not in the domain, so use $x = 2$.

$$m'' = -24x$$
$$m''(2) = -48$$

so the function is concave down at that point so that is a place where the slope is maximized. Therefore the maximum slope is $m(2) = 48(2) - 4(8) = 96 - 32 = 64$

2. Find constants a and b so that $f(x) = a + bx - x^2$ has a local maximum at $(-1, 2)$.

$$f'(x) = b - 2x$$

If there is a maximum at $(-1, 2)$ then

$$f'(-1) = b + 2 = 0 \Rightarrow b = -2$$
$$f(-1) + 2 = a - 2(-1) - (-1)^2 = a + 2 - 1 = a + 1$$
$$2 = a + 1$$
$$a = 1$$

3. Find the dimensions of the rectangle with area 100 cm whose perimeter is minimized.

$$A = xy = 100 \Rightarrow x = \frac{100}{y}$$
$$P = 2x + 2y = 2\left(\frac{100}{y}\right) + 2y$$
$$P = 200y^{-1} + 2y$$
$$P' = -\frac{200}{y^2} + 2 = \frac{-200 + 2y^2}{y^2}$$

Therefore P' dne at $y = 0$ and $P' = 0$ where

$$200 = 2y^2$$

$$y^2 = 100$$
$$y = \pm 10$$

$y \neq -10$

$$P'' = \frac{400}{y^3} > 0 \text{rm at } y = 10$$

so this is a minimum. The dimensions with minimal perimeter are $x = 10$ and $y = 10$.

4. A cylindrical bottle is to be designed so it holds 100 cubic centimeters of perfume. Find the dimensions of the bottle so the amount of material used in the sides and the bottom is as small as possible. (Hint: A cylinder of height h and radius r has volume $\pi r^2 h$, lateral surface area $2\pi rh$, and surface area of top or bottom πr^2.)

$$A = \pi r^2 h = 100 \Rightarrow h = \frac{100}{\pi r^2}$$
$$SA = 2\pi rh + \pi r^2 = 2\pi r \left(\frac{100}{\pi r^2}\right) + \pi r^2 = 200r^{-1} + \pi r^2$$
$$SA' = -\frac{200}{r^2} + 2\pi r = \frac{-200 + 2\pi r^3}{r^2}$$

This dne at $r = 0$, and is zero when

$$200 = 2\pi r^3$$
$$100 = \pi r^3$$
$$\frac{100}{\pi} = r^3$$
$$r = \sqrt[3]{\frac{100}{\pi}}, h = \frac{100}{\pi \left(\sqrt[3]{\frac{100}{\pi}}\right)^2}$$

$$SA' = -200r^{-2} + 2\pi r$$
$$SA'' = \frac{400}{r^3} + 2\pi > 0 \text{ at } r = \sqrt[3]{\frac{100}{\pi}}$$

therefore it really is a minimum there.

5. Suppose the bottle from the previous problem is to be a rectangular box with square base in shape. Find the dimensions of the bottle which minimizes the amount of material used in the sides and bottom. Which shape uses less material?

$$A = x^2 y = 100 \Rightarrow y = \frac{100}{x^2}$$
$$SA = x^2 + 4xy = x^2 + 4x\left(\frac{100}{x^2}\right) = x^2 + \frac{400}{x} = x^2 + 400x^{-1}$$
$$SA' = 2x - \frac{400}{x^2} = \frac{2x^3 - 400}{x^2}$$

This dne if $x = 0$ and is zero if

$$2x^3 = 400 \Rightarrow x^3 = 200 \Rightarrow x = \sqrt[3]{200}$$

Then $y = \frac{100}{\left(\sqrt[3]{200}\right)^2}$

$$SA'' = 2 + \frac{800}{x^3} > 0$$

so it is concave up so it is a minimum.

For the cylinder

$$SA = 2\pi\sqrt[3]{\frac{100}{\pi}}\frac{100}{\pi\left(\sqrt[3]{\frac{100}{\pi}}\right)^2} + \pi(\left(\sqrt[3]{\frac{100}{\pi}}\right)^2 \approx 94.66$$

For the rectangular SA, check the amount of surface area.

Therefore less material is required for the cylinder.

6. You are in charge of security for a concert. You have 43 security guards to be placed at 25 foot increments around 3 sides of the rectangular audience area. If the fire marshal declares that the density of the audience can be no more than 25 people per 100 square feet, what is the maximum number of people who can attend the concert?

$$\frac{2x}{25} + \frac{y}{25} = 43$$
$$2x + y = 1075$$
$$1075 - 2x = y$$
$$A = xy = x(1075 - 2x) = 1075x - 2x^2$$
$$A' = 1075 - 4x$$
$$A' = 0 \text{ when } 1074 = 4x \Rightarrow x = \frac{1075}{4}$$
$$A'' = -4$$

so this is a maximum.

Then

$$y = 1075 - \frac{2150}{4} = 1075 - \frac{1075}{2} = \frac{1075}{2}$$

So the dimensions are

$$\frac{1075}{4} \text{ by } \frac{1075}{2} = 268.75 \times 537.5$$

and the area is

$$A = \frac{1075}{4} \cdot \frac{1075}{2} = \frac{1075^2}{8} = 144453.125$$

Therefore the maximum capacity is A square feet times 25 people per 100 square feet, so 36113 people.

7. The demand equation for a certain product is $p = 6 - \frac{1}{2}x$ where x is the number of units sold and p is the price. Find the level of production which results in maximum revenue. (Hint: revenue is price times quantity sold.)

$$R = px = 6 - \frac{1}{2}x)x = 6x - \frac{1}{2}x^2$$

$$R' = 6 - x$$

This is zero if

$$6 - x = 0$$

$$x = 6$$

$$R'' = -1 < 0$$

so the function is always concave down so $x = 6$ is a minimum. Therefore the production level should be 6.

8. Find the dimensions of a closed rectangular box with square base and volume 800 cubic centimeters which is constructed from the least amount of material.

$$V = x^2 y = 800 \Rightarrow y = \frac{800}{x^2}$$

$$SA = 2x^2 + 4xy = 2x^2 + 4x\left(\frac{800}{x^2}\right) = 2x^2 + \frac{3200}{x}$$

$$SA' = 4x - \frac{3200}{x^2} = \frac{4x^3 - 3200}{x^2}$$

$$SA' = 0 \text{ when } 4x^3 = 3200$$

$$x^3 = 800$$

$$x = \sqrt[3]{800} = 2\sqrt[3]{100}$$

$$SA'' = 4 + \frac{6400}{x^3} > 0$$

at the point we care about, so the function is concave up so there is a minimum.

Then the dimensions are

$$2\sqrt[3]{100} \times \frac{400}{\sqrt[3]{100}} \approx 9.28 \times 86.17$$

9. A rancher will make 2 corrals from 215 meters of fence. One corral is a square. the other corral is a rectangle with length 1.5 times its width. What are the dimensions of each corral resulting in the greatest combined area?

Let x be the length of a side of the square. Let y be the length of the shorter side of the rectangle.

$$A = x^2 + 1.5y^2$$

$$P = 4x + 5y = 215$$

$$4x = 215 - 5y$$

$$x = \frac{215 - 5y}{4}$$

$$A = \left(\frac{215}{4} - \frac{5}{4}y\right)^2 + 1.5y^2$$

$$A' = 2\left(\frac{215}{4} - \frac{5}{4}y\right)\left((-\frac{5}{4}\right) + 3y$$

$$= \left(\frac{215}{2} - \frac{5}{2}y\right)\left(-\frac{5}{4}\right) + 3y$$
$$= -\frac{1075}{8} + \frac{25}{8}y + 3y$$

So $A' = 0$ when

$$\frac{1075}{8} = \frac{49}{8}y$$
$$y = \frac{1075}{49}$$

So

$$x = \frac{215 - \frac{5\cdot1075}{49}}{4} = \frac{215}{4} = \frac{5 \cdot 1075}{4 \cdot 49} \approx 26.32$$

Therefore the side of the square should be about $26.32m$. and the dimensions of the rectangular pen should be approximately $21.94 \times 32.91m$.