Math 142
Spring, 2006 Name: **Solutions**

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1. Compute the following limits:

(a) $\lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} &= \lim_{h\to 0} \frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-x-h}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= -\frac{1}{x\cdot x} = \frac{-1}{x^2}
\end{aligned}$$

(b) $\lim_{h\to 0} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h} \left(\frac{\sqrt{h^2+4h+5}+\sqrt{5}}{\sqrt{h^2+4h+5}+\sqrt{5}}\right) \\
&= \lim_{h\to 0} \frac{h^2+4h+5-5}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)} \\
&= \lim_{h\to 0} \frac{h^2+4h}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)} \\
&= \lim_{h\to 0} \frac{h+4}{\sqrt{h^2+4h+5}+\sqrt{5}} \\
&= \frac{4}{\sqrt{5}+\sqrt{5}} \\
&= \frac{4}{2\sqrt{5}} \\
&= \frac{2}{\sqrt{5}}
\end{aligned}$$

(c) $\lim_{x\to 2} \frac{|\,x-2\,|}{x-2}$

$$\lim_{x\to 2^+} \frac{|\,x-2\,|}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2}$$
$$= \lim_{x\to 2^+} 1 = 1$$

$$\lim_{x\to 2^-} \frac{|\,x-2\,|}{x-2} = \lim_{x\to 2^-} \frac{-(x-2)}{x-2}$$
$$= \lim_{x\to 2^+} -1 = -1$$

Therefore the limit does not exist (since the two one-sided limits do not match)

(d) $\lim_{x\to 3} \sqrt{\frac{x+2}{x+1}}$

$$\lim_{x\to 3} \sqrt{\frac{x+2}{x+1}} = \sqrt{\lim_{x\to 3} \frac{x+2}{x+1}}$$

Since this is a rational function which is defined at $x = 3$ we can just plug in $x = 3$:

$$= \sqrt{\frac{3+2}{3+1}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

(e) $\lim_{x\to 0^+} \sqrt{x}\sin\left(\frac{1}{x}\right)$

Use the sandwich theorem:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-\sqrt{x} \leq \sqrt{x}\sin\left(\frac{1}{x}\right) \leq \sqrt{x}$$

As $x \to 0^+$ of each part of this gives;

$$0 \leq \lim_{x\to 0} \sqrt{x}\sin\left(\frac{1}{x}\right) \leq 0$$

Therefore by the sandwich theorem, we must have that

$$\lim_{x\to 0} \sqrt{x}\sin\left(\frac{1}{x}\right) = 0$$

(f) $\lim_{x\to\infty} \frac{x^7-1}{x^6+1}$

$$\lim_{x\to\infty} \frac{x^7-1}{x^6+1} = \lim_{x\to\infty} \frac{\frac{1}{x^6}\left(x^7-1\right)}{\frac{1}{x^6}\left(x^6-1\right)}$$
$$= \lim_{x\to\infty} \frac{x-\frac{1}{x^6}}{1+\frac{1}{x^6}}$$
$$= \infty$$

(g) $\lim_{x\to\infty} \frac{x-1}{\sqrt{x^2+29}}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}(x-1)}{\frac{1}{x}\sqrt{x^2+29}}$$

$$= \lim_{x\to\infty} \frac{1-\frac{1}{x}}{\frac{1}{\sqrt{x^2}}\sqrt{x^2+29}}$$

$$= \lim_{x\to\infty} \frac{1-\frac{1}{x}}{\sqrt{1-\frac{29}{x^2}}}$$

$$= \frac{1}{\sqrt{1}}$$

$$= 1$$

(h) $\lim_{x\to\infty} \frac{3x^{29}+7}{3x^{29}+5}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x^{29}}\left(3x^{29}+7\right)}{\frac{1}{x^{29}}\left(3x^{29}+5\right)}$$

$$= \lim_{x\to\infty} \frac{3+\frac{7}{x^{29}}}{3+\frac{5}{x^{29}}} = \frac{3}{3} = 1$$

2. Which of the above functions (in (a) thru (e)) are continuous at the points where you evaluated the limits?

   Notice that no function can be continuous at infinity because we can't evaluate the function at that point.

   (a) This is not continuous since $h = 0$ is not in the domain of the function.
   (b) This is not continuous since $h = 0$ is not in the domain of the function.
   (c) The limits don't match (and the point is not in the domain)
   (d) This is continuous
   (e) This is not continuous since $\frac{1}{x}$ is not defined at $x = 0$