RESEARCH STATEMENT

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My research has been focused in two areas - the first of which was the topic of my doctoral dissertation and the second of which has evolved over the last several years. My doctoral dissertation was concerned with the homotopy classification of 2-dimensional CW-complexes. This typically involves techniques from algebraic and combinatorial topology as well as homological and combinatorial group theory. A good overview of what is happening in this field can be found in the book "Two-Dimensional Homotopy and Combinatorial Group Theory" [HAMS]. My more recent research activity has been focused on knot theory, and more specifically on knots where the canonical genus of the Whitehead double of the knot is equal to the crossing number of the undoubled knot. We will begin with a discussion of the questions we have examined in the area of 2-complexes.

1. Homotopy Classification of 2-Dimensional CW-Complexes

It is well known that every 2-dimensional CW-complex, hereafter 2-complex, is homotopy equivalent to a 2-complex built as the standard 2-complex of a presentation, $\mathcal{P} = \langle \underline{\underline{x}} | \underline{\underline{r}} \rangle$, of a group, G. This is done using a single 0-cell, attaching a 1-cell for each $x \in \underline{\underline{x}}$, and a 2-cell for each $r \in \underline{\underline{r}}$ by glueing the boundary to the 1-skeleton, spelling out the relator word. A 2-complex built in this way will have the group G as its fundamental group. The question then becomes: Can two standard 2-complexes with the same Euler characteristic, built from presentations of the same group, be homotopy inequivalent?

It is easy to see that there is a minimal Euler characteristic for each group, $\chi_{\min}$. Any presentation can be made to have a higher Euler characteristic by adding a trivial relator, 1, which is equivalent to taking the previous complex, $X_{\mathcal{P}}$, and forming $X_{\mathcal{P}} \vee S^2$. If $X_{\mathcal{P}}$ had Euler characteristic $\chi_{\min}$, then this new complex has Euler characteristic $\chi_{\min} + 1$.

Restricting our attention to standard 2-complexes arising from two presentations of the same group having the same Euler characteristic, the question of the number of homotopy types has been answered in many cases. If G is a free group, there is a unique homotopy type at every Euler characteristic level [HAMS, Chapter 3]. If G is a finite group, there may be multiple homotopy types at the minimal level, but according to a theorem of Browning [Br], there is a unique homotopy type above $\chi_{\min}$.

Therefore all of these examples of homotopy inequivalent 2-complexes seem to occur at the level of the minimal Euler characteristic. The only previously known example of homotopy inequivalent 2-complexes laying above the minimal level was

Dunwoody's new presentation of the trefoil knot. The standard presentation for the trefoil knot is $\mathcal{T} = \langle a, b | a^2 b^{-3} \rangle$. Dunwoody and Pietrowski [DP] discovered that the presentation

$$\mathcal{D} = \langle x, y | (x^2y^{-3})(x^2y^{-3})^x(x^2y^{-3})^{x^2}, (x^2y^{-3})(x^2y^{-3})^y(x^2y^{-3})^{y^2}(x^2y^{-3})^{y^3} \rangle$$

where $w^x = x^{-1}wx$, is also a presentation for the trefoil knot group. Dunwoody [D] further proved that for the standard 2-complex of this presentation, $\pi_2 X_{\mathcal{D}} \cong R/[R, R]$, where R is the normal closure of the relator words, and $[R, R]$ is the commutator subgroup. This allowed Dunwoody to show [D2] that $\pi_2 X_{\mathcal{D}}$ requires at least two generators. We can then compare this presentation to the presentation $\mathcal{T} = \langle a, b | a^2 b^{-3}, 1 \rangle$, the standard presentation of the trefoil knot group occurring at level $\chi_{\min} + 1$. Considering the standard 2-complex of this presentation we see that it requires only one generator for π_2. Therefore, these two presentations of the trefoil knot group give rise to homotopy inequivalent 2-complexes, and this example lies at $\chi_{\min} + 1$.

Dunwoody's method can be extended to always develop a presentation for the second homotopy group. This method is detailed in [J]. This method, however, has not been as helpful as desired in producing new examples of homotopy inequivalent 2-complexes at non-minimal levels. However, joint work with Jens Harlander has been more fruitful. We have discovered another generalization of Dunwoody's techniques which provide examples for each n of $n-$dimensional groups G for which there exist homotopically distinct $(G, n)-$complexes on the level $\chi_{\min}(G, n) + 1$, with the examples at level $n = 2$ all being algebraic. For more details, see [HJ2].

2. Canonical Genus and Whitehead Doubles of Knots

More recently, my research emphasis has been on knot theory, specifically on the examination of the canonical genus of the Whitehead doubles of knots. Every knot K in the 3-sphere S^3 is the boundary of a compact orientable surface $\Sigma \subset S^3$, known as a Seifert surface for the knot K. The first proof of this was given by Seifert [S], who gave an algorithm which starts with a diagram of the knot K and produces such a surface. The algorithm consists of orienting the knot diagram, breaking each crossing and reconnecting the resulting four ends according to the orientation, without re-introducing a crossing, producing disjoint Seifert circles in the projection plane, bounding (after offsetting nested circles) disjoint Seifert disks, and then reintroducing the crossings of K by stitching the disks together with half-twisted bands.

The minimum of the genera of the canonical surfaces built by Seifert's algorithm, over all diagrams of the knot K, is known as the canonical genus of K, denoted $g_c(K)$. The minimum genus over all Seifert surfaces, whether built by Seifert's algorithm or not, is known as the genus of K, and denoted $g(K)$.

Recently, Morton's inequality, which relates the degree of the HOMFLY polynomial to the canonical genus of knots, and will be discussed further below, has been applied to the computation of the canonical genus of the Whitehead doubles $W(K)$ of certain knots. Tripp [Tr] posed the question whether the canonical genus of the Whitehead doubles of knots is equal to the crossing number of the original knot. This has been shown to be true in the case of torus knots, [Tr], and 2-bridge knots, [Na]. Joint work with Mark Brittenham extends these results to show that the canonical genus of the Whitehead double of many other reduced prime alternating knots is equal to the crossing number of the original knots.

The main result of one paper on this topic [BJ] is the following:

Theorem. *Let L be a non-split link with a diagram D' satisfying $c(D') = c(L)$ and the maximum $z-$degree of the HOMFLY polynomial of the flat Whitehead double being $2c(D') - 1$. If K is a link having diagram D obtained by replacing a crossing in the diagram D' with a full twist (so that $c(D) = c(D') + 1$), then the maximum $z-$degree of the HOMFLY polynomial of the flat Whitehead double of K is $2c(D) - 1$.*

This immediately implies the following corollary:

Theorem. *If K is a pretzel knot $P(k_1, \dots k_n)$ with $k_1, \dots, k_n \geq 1$, then $g_c(W(K)) = k_1 + \cdots + k_n = c(K)$.*

We, therefore, get a family of classes of knots for which the conjecture holds.

We continued our study of this area, also investigating situations in [BJ2] in which the HOMFLY polynomial is not sufficient to examine the canonical genus of the knot. In 1985, shortly after the discovery of the HOMFLY polynomial [FHLMOY], Morton [Mo] showed that the highest degree of one of the two variables gave a lower bound on $2g_c(K)$. This was perhaps the first piece of information encoded in the HOMFLY polynomial to be related to topological information about the knot K. Morton's inequality, $M(K) = \max \deg_z P_K(v, z) \leq 2g_c(K)$, has since been shown to be an equality for many classes of knots. These include all of the knots having 12 or fewer crossings [St], all alternating knots [Cr], [Mu], and, more generally, all homogeneous knots [Cm], and the Whitehead doubles of 2-bridge knots [Na], [Tr] and pretzel knots [BJ]. It wasn't until 1998 and later that knots were found for which Morton's inequality was strict. The first such were found by Stoimenow [St],[St2] while analyzing the survey of knots through 16 crossings.

We have shown how to use Stoimenow's second collection of examples, or any other example that might be built along the same lines, to build infinite families of knots having

$$M(K) = \max \deg_z P_K(v, z) < 2g_c(K).$$

In particular, we have the following:

Theorem. *Suppose K is a knot with $g(K) = g_c(K)$ and $M(K) < 2g_c(K)$. Let D be a g_c-minimizing diagram of K, and suppose there is a crossing c in D which bounds a half-twisted band connecting a pair of Seifert disks, and the knot K' obtained by changing the crossing c has $g_c(K') < g_c(K)$. Then the knots K_n,* *bounding the canonical Seifert surfaces Σ_n obtained by replacing the half-twisted band at the crossing c with $2n+1$ half-twisted bands in parallel, all joining the same pair of Seifert disks, all satisfy $M(K_n) < 2g_c(K_n) = 2g_c(K) + 2n$.*

3. Future Work

It is now reasonable to ask "For what other (classes of) knots does $g_C(W(K)) = c(K)$ hold?" We know that the techniques that we employ will not establish this result for all alternating knots, since the equality max $\deg_z(P_{W(K)}(v,z) = c(K)$ might not hold for non-prime alternating knots. However, this does not show that $g_C(W(K)) = 2c(K)$ does not hold. However, based on our investigations, we feel it is reasonable to make the following conjecture:

Conjecture. *If K is a nontrivial prime alternating knot and W(K) is a Whitehead double of K, then max $\deg_z(P_{W(K)}(v,z)) = 2c(K)$ and therefore $g_c(W(K)) = c(K)$.*

We are also seeking non-alternating knots to which our methods apply, ie., where

$$\max \text{def}_z(P_{W(K)}(v,z)) = 2c(K).$$

The results we have already developed would then show that these knots form the basis of other collections of inifinite families of examples.

Further, we wonder if there exists a non-trivial knot K whose Whitehead doubles have different canonical genera? Of course, our methods will not be able to locate these examples, since their HOMFLY polynomials will all have the same $z-$degree. However, these examples could also be interesting to locate.

References

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