16. If R is any ring, then the center of the ring R is the set $C = \{c \in R \mid cr = rc \ \forall r \in R\}$.

    (a) Show C is a subring of R.

    (b) A subring I of R is a left ideal provided $r \in R$ and $x \in I \Rightarrow rx \in I$. A subring I of R is a right ideal provided $r \in R$ and $x \in I \Rightarrow xr \in I$. I is an ideal if it is both a left and right ideal. Show that C may not be an ideal.

17. An element r in a ring R is idempotent if $r^2 = r$. If $r \in R$ is an idempotent element and $x \in R$ then $(xr - rxr)^2 = 0_R$.

18. An element a of a ring R is nilpotent if $a^n = 0$ for some positive integer n.

    (a) Prove that in a commutative ring $a + b$ is nilpotent if a and b are.

    (b) Show that this result may be false if R is not commutative.

    (c) Prove that R has no nonzero nilpotent elements if and only if 0_R is the unique solution of the equation $x^2 = 0_R$.

19. Let R be a ring in which $x^3 = x \forall x \in R$. Prove that R is commutative.

20. Let G be a group. If $a, b, c \in G$, prove that there is a unique element $x \in G$ such that $axb = c$.

21. Let G be a group. If $a, b \in G$ with $b^6 = e$ and $ab = b^4a$, prove that $b^3 = e$ and $ab = ba$.

22. Let G be a group. If $(ab)^n = a^n b^n$ for three consecutive integers n and for all $a, b \in G$, prove that G is abelian.

23. Let G be a nonempty set equipped with an associative operation with the properties

    i. There is an element $e \in G$ such that $ea = a$ for all $a \in G$

    ii. For each $a \in G$ there exists $d \in G$ such that $da = e$.

    Show that G is a group.

24. Let G be a cyclic group. Prove that every subgroup of G is cyclic.

25. Prove that if G is a cyclic group, then G is abelian.

26. Prove that if $G = \langle a \rangle$ is a cyclic group then $G = \{a^n \mid n \in \mathbb{Z}\}$.