

IMMERSE Algebra Homework - Summer 2005  
Normal Subgroups

62. Let  $G$  be a group with  $H < G$ .
- (a) Prove that the relation on  $G$  defined by  $a \sim b \iff b^{-1}a \in H$  is an equivalence relation on  $H$ .
  - (b) Show that for any element  $a \in H$ , the equivalence class of  $a$ , under the relation given above) is  $aH = \{ah \mid h \in H\}$ .
63. If  $\Sigma_3$ , let  $a = (12)$ .
- (a) Compute  $aC_3a^{-1}$
  - (b) Prove that there is an element  $h \in C_3$  such that  $aha^{-1} \neq h$ .
  - (c) Is  $C_3 \triangleleft \Sigma_3$ ? Prove or disprove.
64. Let  $H < G$ . Prove  $|aH| = |H| \quad \forall a \in G$ .
65. Give an example of a group  $G$ , a subgroup  $H$  and an element  $a \in G$  such that  $aHa^{-1} \subset H$ , but  $aHa^{-1} \neq H$ .
66. Find an example of a group  $G$  and a subgroup  $H$  where the natural “multiplication” on  $\overline{G} = \{aH \mid a \in G\}$  is not well-defined.
67. Prove or Disprove: If  $M \triangleleft N$  and  $N \triangleleft G$  then  $M \triangleleft G$ .
68. Let  $N < \Sigma_4$  consist of all those permutations  $\sigma$  such that  $\sigma(4) = 4$ . Is  $N$  normal in  $\Sigma_4$ ? Prove or disprove.
69. Let  $G$  be a group. A commutator in  $G$  is an element of the form  $aba^{-1}b^{-1}$  with  $a, b \in G$ . Let  $G^C$  be the subgroup generated by the commutators. Then  $G^C$  is called the commutator subgroup. Show that  $G^C$  is normal.
70. If a cyclic subgroup  $T$  of  $G$  is normal in  $G$ , then show that every subgroup of  $T$  is normal in  $G$ .
71. Let  $H$  be a subgroup of  $G$ . If  $\langle a \rangle \triangleleft G \quad \forall a \in H$  then  $H \triangleleft G$ .