IMMERSE Algebra Homework - Summer 2005
Hamiltonian Groups

72. Let G be a finite group of even order, and let $H < G$ with $\mid H \mid = \frac{1}{2} \mid G \mid$. Prove $H \triangleleft G$.

73. Let H be a subgroup of order n in a group G. If H is the only subgroup of order n then H is normal in G.

74. Let G be a group, $H < G$, $K < G$. Define $HK = \{hk \mid h \in H, k \in K\}$.

    (a) Show $HK < G$ iff HK is closed under the binary operation of the group.

    (b) Show that if $H \triangleleft G$ or $K \triangleleft G$ then $HK < G$.

    (c) Show $\mid HK \mid = \frac{|H||K|}{|H \cap K|}$

    (d) Let $H = \langle \sigma \rangle$ and $K = \langle \tau \rangle$ where σ is a 3-cycle and τ is a 2-cycle in Σ_3. Show $HK = \Sigma_3$.

75. Suppose H and K are normal subgroups of a group G with $H \cap K = \langle e \rangle$. Prove $hk = kh$ for all $h \in H$, $k \in K$.

76. Let $\mathbb{C}^*$ be the group of non-zero complex numbers under multiplication. Geometrically, we can identify $a + bi \in \mathbb{C}^*$ with the point (a, b) in the cartesian coordinate plane. Let $H = \{a + bi \mid a^2 + b^2 = 1\} < \mathbb{C}^*$. Give a geometric description of the cosets of H.

77. Show that every Abelian group satisfies the hamiltonian group property (i.e. every subgroup is normal).

78. Show that the group of quaternions is a hamiltonian group. (note: the quaternions are the smallest hamiltonian group).

79. Show that if G is a non-abelian group such that every cyclic subgroup of G is normal in G, then G is hamiltonian.