123. Find a polynomial $p(x) \in \mathbb{Q}[x]$ so that

$$\mathbb{Q}\left(\sqrt{4+\sqrt{14}}\right) \approx \mathbb{Q}[x] / \langle p(x) \rangle .$$

124. Let p be a prime, $n \in \mathbb{Z}^+$. Let $f(x) = x^{p^n} - x \in \mathbb{Z}_p[x]$. Let E be a splitting field of $f(x)$. Let $F = \{a \in E : f(a) = 0\}$. Prove F is a subfield of E.

125. Let $f(x) \in F[x]$ (where F is a field). Show that $f(x)$ has a multiple root a if and only if a is a root of $D_x(f(x))$.

126. Prove $[\mathbb{F}_{p^n} : \mathbb{F}_p] = n$ for any prime p and any positive integer n.

127. Let p be a prime and $m, n \in \mathbb{Z}^+$ with $m < n$.

   (a) Show that if m divides n, then $\mathbb{F}_{p^m}$ is a subfield of $\mathbb{F}_{p^n}$.

   (b) Show that if m does not divide n, then $\mathbb{F}_{p^m} \cap \mathbb{F}_{p^n}$ contains fewer than p^m elements, so $\mathbb{F}_{p^m}$ is not a subfield of $\mathbb{F}_{p^n}$.

128. Prove $x^2 + x + 1 = 0$

   (a) has solutions in $\mathbb{F}_4$.

   (b) has solutions in $\mathbb{F}_{2^n}$ for all n even.

   (c) has no solutions in $\mathbb{F}_{2^n}$ for all n odd.

129. Let p be an odd prime. Prove $x^2 + y^2 + 1 = 0$ has a solution in $\mathbb{Z}_p$.

130. Let R be a finite commutative ring with identity.

   (a) Prove every element of R is either a zero-divisor or a unit.

   (b) Prove every finite integral domain is a field.