118. Prove every finite (multiplicative) subgroup of F^* is cyclic (where F is a field and $F^* = F - \{0\}$). So, if F is a finite field, then F^* is a cyclic group.

119. Let F be a field, E be an extension field of F and $a \in E$. Show $F[a]$ exists by showing

$$S = \bigcap \{K : K \text{ is a subring of } E \text{ and } K \text{ contains } F \text{ and } a\}$$

satisfies the definition of $F[a]$ (so $F[a] = S$). i.e.

(a) S is a ring,

(b) $F \subseteq S$ and $a \in S$, and

(c) if R is a subring of E and $F \subseteq R$ and $a \in R$, then $S \subseteq R$.

120. Complete the proof that

$$F[a] = \left\{ \sum_{i=0}^{n} r_i a^i : n \in \mathbb{Z}^+ \text{ and } r_i \in F \right\} = \{f(a) : f(x) \in F[x]\}.$$

121. Prove the following:

(a) $\mathbb{Q}\left[\sqrt{2}\right] = \left\{a + b\sqrt{2} : a, b \in \mathbb{Q}\right\}$ (using the **definition** of $F[a]$)

(b) $\mathbb{Q}\left[\sqrt{2}\right] = \mathbb{Q}\left(\sqrt{2}\right)$

(c) Determine $\left[\mathbb{Q}\left[\sqrt{2}\right] : \mathbb{Q}\right]$.

122. Let $\alpha = \sqrt[3]{2 + \frac{10}{9}\sqrt{3}} + \sqrt[3]{2 - \frac{10}{9}\sqrt{3}}$.

(a) Verify that $\left(1 + \frac{1}{3}\sqrt{3}\right)^3 = 2 + \frac{10}{9}\sqrt{3}$.

(b) Find $a, b \in \mathbb{Q}$ such that $\left(a + b\sqrt{3}\right)^3 = 2 - \frac{10}{9}\sqrt{3}$.

(c) Prove $\alpha = 2$.

(d) Explain why $\mathbb{Q}\left(\sqrt[3]{2 + \frac{10}{9}\sqrt{3}} + \sqrt[3]{2 - \frac{10}{9}\sqrt{3}}\right) = \mathbb{Q}$.

(e) Determine $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.