IMMERSE Algebra Homework - Summer 2005
Field Theory I

112. Let E be an extension field of the field F. Show that E is a vector space over F.

113. Construct a field of order 7^3 of the form $\mathbb{Z}_n[x]/\langle f(x)\rangle$.

114. Let $p(x) \in F[x]$ where F is a field.

    (a) Show that if $f(x) \in \langle p(x)\rangle$, then $f(x) = 0$ or $\deg f(x) \geq \deg p(x)$.

    (b) In $\mathbb{Z}_{12}[x]$, find $f(x) \in \langle p(x)\rangle$ where $p(x) = 4x + 1$ such that $\deg f(x) < \deg p(x)$. (relate this to part (a)).

    (c) Show for $f(x), g(x) \in F[x]$, if $\deg f(x) < \deg p(x)$ and $\deg g(x) < \deg p(x)$, then $f(x) + \langle p(x)\rangle = g(x) + \langle p(x)\rangle$ if and only if $f(x) = g(x)$.

    (d) Prove if $f(x) + \langle p(x)\rangle \in F[x]/\langle p(x)\rangle$, then $f(x) + \langle p(x)\rangle = g(x) + \langle p(x)\rangle$ for some $g(x) \in F[x]$ with $g(x) = 0$ or $\deg g(x) < \deg p(x)$.

    (e) From parts (c) and (d) of this problem we can conclude that every element of $F[x]/\langle p(x)\rangle$ can be written uniquely in the form $g(x) + \langle p(x)\rangle$ where $g(x) = 0$ or $\deg g(x) < \deg p(x)$. So, if $|F| < \infty$, then $|F[x]/\langle p(x)\rangle| = |F|^{\deg p(x)}$. Use this to verify that the field you constructed in #113 really does have 7^3 elements.

115. Let F be a field and let $f(x) \in F[x]$ be irreducible over F. Show $\{a + \langle f(x)\rangle : a \in F\}$ is a subfield of $F[x]/\langle f(x)\rangle$ isomorphic to F.

116. Let $p(x)$ be irreducible in $F[x]$. Let E be an extension field of F that contains a root, a, of $p(x)$. Define $\varphi_a : F[x] \to E$ by $\varphi_a(f(x)) = f(a)$. Prove φ_a is a homomorphism (non-trivial) with $\ker \varphi_a = \langle p(x)\rangle$.

117. Let V be a vector space over a field F. Prove that every basis for V over F has the same number of elements. You may use the following facts if you find them helpful:

    - Every linearly independent subset of V can be extended to a basis.
    - Every spanning set for V contains a basis.