IMMERSE Algebra Homework - Summer 2005
Characteristic of a Ring and Exponent of a Group

87. Let p be a prime and R a commutative ring with identity of characteristic p.
    (a) Then for every $a, b \in R$ and every positive integer n, $(a+b)^p = a^p + b^p$.
    (b) Show that this may be false if the characteristic p is not prime or if R is not commutative.

88. Let R be a commutative ring with identity of prime characteristic p. If $a, b \in R$ and $n \geq 1$, prove that $(a-b)^{p^n} = a^{p^n} - b^{p^n}$.

89. Let $R = \{(a, b, c) \mid a \in \mathbb{Z}_2, b \in \mathbb{Z}_3, c \in \mathbb{Z}\}$. Define addition and multiplication coordinate-wise. What is the characteristic of R?

90. Give an example of an integral domain which has an infinite number of elements, yet is of nonzero characteristic.

91. Show that $e = \frac{1}{2}(1 + (-1))$ is a central idempotent of $K[Q_8]$ *where* K *is a field of characteristic different from 2.*

92. If R is a ring with unity, 1, then $char(R) = \circ(1)$ if $\circ(1)$ is finite and $char(R) = 0$ if $\circ(1)$ is infinite. (where $\circ(1)$ is the additive order of the multiplicative identity 1).

93. Let F be a field of characteristic p. Prove that the mapping $\phi : \mathbb{Z} \to F$ given by $\phi(n) = n \cdot 1_F$ is a ring homomorphism and that $\ker \phi = p\mathbb{Z}$.

94. Let R be a commutative ring with no non-zero zero divisors.
    (a) Let a and b be nonzero elements of R. Prove that for any integer k, $k \cdot a = 0 \iff k \cdot b = 0$.
    (b) Let a and b be nonzero elements of R. Prove $\circ(a) = \circ(b)$.
    (c) Suppose $char(R) \neq 0$. Let a be a nonzero element of R. Prove $char(R) = \circ(a)$.
    (d) Show $char(R)$ is 0 or prime.
    (e) If R is an integral domain and if $na = 0_R$ for some $a \neq 0$ in R and some integer $n \neq 0$, prove that $char(R)$ is finite.