Minimal Cohen–Macaulay Deformations of Matroid Ideals

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Let $M = \langle m_1, m_2, \ldots, m_r \rangle$ be a monomial ideal in $S = S = \mathbb{k}[x_1, x_2, \ldots, x_n]$.

Let $X$ be a finite regular CW-complex with $r$ vertices.

Label each vertex of $X$ by the generators of $M$, and each face of $X$ by the lcm of the labels of its vertices.

Fix an orientation on $X$.

The **Cellular complex** $\mathbb{F}_X$ supported on $X$ is the complex of $\mathbb{Z}^n$–graded modules

$$
\mathbb{F}_X = \bigoplus_{F \in X} S[-a_F],
$$

$$
\partial(F) = \sum_{\text{facets } G \text{ of } F} \text{sign}(G, F) \frac{x^{a_F}}{x^{a_G}} G
$$
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Let $\mathbb{F}_X$ be the complex of $\mathbb{Z}^n$-graded free $S$-modules

$$
0 \rightarrow S[-(2, 2, 1)] \oplus S[-(1, 2, 2)] \xrightarrow{\partial_2} S[-(2, 1, 1)] \oplus S[-(2, 2, 0)] \oplus
S[-(1, 2, 1)] \oplus S[-(1, 1, 2)] \oplus S[-(0, 2, 2)] \xrightarrow{\partial_1} S[-(2, 1, 0)] \oplus
S[-(1, 0, 1)] \oplus S[-(0, 1, 2)] \oplus S[-(0, 2, 0)] \xrightarrow{\partial_0} S
$$
The differential $\partial$ acts on basis vectors

$$\partial(a^2b^2c) = -b \cdot a^2bc + c \cdot a^2b^2 - a \cdot ab^2c$$

Observe $\partial_0 = \begin{bmatrix} a^2b & ac & bc^2 & b^2 \end{bmatrix}$. Thus $\text{Coker}(\partial_0) = S/M$. 
For $b \in \mathbb{N}^n$, let $X_{\leq b}$ be the subcomplex of $X$ consisting of all faces whose degrees are coordinatewise at most $b$.

**Theorem.** $\widetilde{F}_X$ is exact if and only if $X_{\leq b}$ is acyclic over $k$ for all $b \in \mathbb{N}^n$. 
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**Theorem.** $\mathbb{F}_X$ is exact if and only if $X_{\leq b}$ is acyclic over $k$ for all $b \in \mathbb{N}^n$.

**Example.** Let $b = (1, 1, 2)$, then $X_{\leq b}$ is the subcomplex

\[
\begin{array}{cc}
ac & abc^2 \\
\downarrow & \downarrow \\
bc^2
\end{array}
\]

$(\mathbb{F}_X)_b$ equals the reduced chain complex

\[
\tilde{C}(X_{\leq b}; k) = 0 \rightarrow k \rightarrow k^2 \rightarrow k
\]

$X_{\leq b}$ is contractible, so it has no reduced homology.
Let $\mathcal{M}$ be a matroid on the set $\{1, \ldots, n\}$, and let $L$ be its lattice of flats.

For every proper flat $F \in L$ let $m_x(F) = \prod_{i : i \notin F} x_i$.

The matroid ideal $M$ is the monomial ideal

$$M = \langle m_x(F) \mid F \text{ is a proper flat} \rangle$$
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A square–free monomial ideal $M$ is a matroid ideal if and only if for every pair of monomials $m_1, m_2 \in M$ and any $i \in \{1, \ldots, n\}$ such that $x_i$ divides both $m_1$ and $m_2$, the monomial $\text{lcm}(m_1, m_2)/x_i$ is in $M$ as well.

\[ M = \langle x_1 x_2, x_1 x_3 x_4, x_1 x_3 x_5, x_2 x_3 x_4, x_2 x_3 x_5, x_4 x_5 \rangle. \]
Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an affine $\ell$-arrangement.

The affine matroid $\Gamma_\mathcal{A}$ on $\{1, \ldots, n\}$ is the set of all subsets $S \subset [n]$ such that $S \cup \{0\}$ is a cocircuit of the matroid associated to the cone $c\mathcal{A}$.

The affine matroid ideal $M_\mathcal{A}$ is generated by the monomials $m_C = x_{i_1}x_{i_2}\cdots x_{i_t}$, where $C = \{i_1, \ldots, i_t\} \in \Gamma_\mathcal{A}$.

If $H_0$ is in general position relative to $\mathcal{A}$ ($\mathcal{A}$ is transverse to the hyperplane at infinity) then $M_\mathcal{A}$ is a matroid ideal.

The ideal $M_\mathcal{A}$ is minimally generated by the monomials $m_x(v) = \prod_{v \notin H_i} x_i$, where $v$ ranges over the vertices of $\mathcal{A}$.

**Theorem.** Let $B_\mathcal{A}$ be the bounded complex of $\mathcal{A}$. Then its cellular complex $C_\bullet(B_\mathcal{A}, M_\mathcal{A})$ gives a minimal free resolution for $M_\mathcal{A}$. 
Let $\mathcal{A}$ be the 2–arrangement

\[
H_1 \quad H_2 \quad H_3 \quad H_4 \quad H_5
\]

\[
x_1 x_2 \quad x_1 x_3 x_4 \quad x_1 x_3 x_5 \quad x_2 x_3 x_5 \quad x_2 x_3 x_4
\]

The bounded complex $B_{\mathcal{A}}$ resolves $S/M_{\mathcal{A}}$ minimally.
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Hence $0 \to S^4 \to S^9 \to S^6 \to S$ is a minimal free resolution for $S/M_\mathcal{A}$.
We say that a monomial $m'$ strictly divides $m$ if $\text{supp}(\frac{m}{m'}) = \text{supp}(m)$.

A monomial ideal $M = \langle m_1, \ldots, m_r \rangle$ is called generic if, whenever two distinct minimal generators $m_i$ and $m_j$ have the same positive degree in some variable $x_s$, there is a third generator $m_l$ which strictly divides $\text{lcm}(m_i, m_j)$.

Example. $M = \langle x^2y^2, x^2z^2, yz \rangle$ is generic.
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For \( \sigma \subseteq \{1, \ldots, r\} \), let \( m_\sigma = \text{lcm}(m_i, i \in \sigma) \).

The **Scarf complex** of \( M \) consists of the following subsets:

\[
\Delta_M := \{ \sigma \subseteq \{1, \ldots, r\} \mid m_\sigma \neq m_\tau \ \forall \ \tau \subseteq [r], \tau \neq \sigma \}.
\]
Let \( M = \langle a^2b, ac, b^2, bc^2 \rangle \). The Scarf complex of \( M \) is

\[
\begin{array}{ccc}
ac & abc^2 & bc^2 \\
\\
a^2bc & ab^2c & b^2c^2 \\
\\
a^2b & a^2b^2 & b^2
\end{array}
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\begin{array}{ccc}
ac & abc^2 & bc^2 \\
\hline
a^2bc & ab^2c^2 & b^2c^2 \\
a^2b^2 & a^2b^2c & b^2c^2
\end{array}
\]

**Theorem.** *For* \( M \) *generic, the cellular complex* \( \bf{F}_{\Delta_M} \) *is a minimal free resolution of* \( S/M \).

The minimal free resolution of \( S/M \) is

\[
0 \to S^2 \to S^5 \to S^4 \to S
\]
Let $M = \langle m_1, m_2, \ldots, m_r \rangle$ be a square–free monomial ideal. A deformation of $M$ is a monomial ideal $M^* = \langle m_1^*, m_2^*, \ldots, m_r^* \rangle$, such that, for all $i \in [r]$, $\text{supp}(m_i^*) = \text{supp}(m_i)$.

Denote by $\Delta_{M^*}$ the Scarf complex of a generic monomial ideal $M^*$. Then, $F_{\Delta_{M^*}}$ is a resolution of $S/M$ after relabeling the vertices of $\Delta_{M^*}$ with the generators of $M$.

Every matroid ideal $M$ is Cohen–Macaulay. But $M^*$ is not necessarily Cohen–Macaulay even if $M^*$ is a generic deformation.
Cohen–Macaulay Generic Deformations

Let $M = \langle m_1, m_2, \ldots, m_r \rangle$ be a square–free monomial ideal. A deformation of $M$ is a monomial ideal $M^* = \langle m_1^*, m_2^*, \ldots, m_r^* \rangle$, such that, for all $i \in [r]$, $\text{supp}(m_i^*) = \text{supp}(m_i)$.

Denote by $\Delta_{M^*}$ the Scarf complex of a generic monomial ideal $M^*$. Then, $F_{\Delta_{M^*}}$ is a resolution of $S/M$ after relabeling the vertices of $\Delta_{M^*}$ with the generators of $M$.

Every matroid ideal $M$ is Cohen–Macaulay. But $M^*$ is not necessarily Cohen–Macaulay even if $M^*$ is a generic deformation.

**Theorem.** Let $M_A$ be a matroid ideal associated to an affine $\ell$–arrangement $A$. Let $M^*_A$ be a generic deformation. Then $\dim(B_A) = \dim(\Delta_{M^*_A})$ if and only if $M^*_A$ is Cohen–Macaulay.
Cohen–Macaulay Generic Matroid Ideals

Matroid ideals always have Cohen–Macaulay generic deformations.

\[ M = \langle x_1x_2, x_1x_3x_4, x_2x_3x_4, x_1x_3x_5, x_2x_3x_5, x_4x_5 \rangle \]

● Fix the order 1, 2, 3, 4, 5 in the indeterminates of \( S \). Then

\[ M^A = \langle x_1x_2, x_1^2x_3x_4, x_2^2x_3x_4, x_3^3x_3x_5, x_2^3x_3x_5, x_4^3x_5^3 \rangle \]

is a CM generic deformation of \( M \).

● \( \Delta_{M^A} \) has facets \( \{1, 2, 4\}, \{1, 3, 5\}, \{1, 2, 6\}, \{1, 3, 6\} \)
Fix the order $1, 3, 4, 2, 5$. Then

$$M^A = \langle x_1 x_3 x_4, x_1^2 x_2, x_2^2 x_3 x_4, x_1^3 x_3^3, x_2^3 x_3 x_5, x_4^3 x_5^3 \rangle$$

is the **CM generic deformation** of $M$ associated to the given order.

$\Delta_{M^A}$ has facets $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 5\}$, $\{1, 2, 6\}$, $\{1, 3, 6\}$
Theorem. Let $\mathcal{A}$ be an affine $\ell$–arrangement transverse to the hyperplane at infinity. Then, there exists a Cohen–Macaulay generic deformation of the matroid ideal $M_\mathcal{A}$ that gives a minimal free resolution of $S/M_\mathcal{A}$ if (and only if) the arrangement $\mathcal{A}$ is supersolvable.