# On The Minimum Vector Rank of a Multigraph

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**Abstract**

Given a multigraph $G$ on the vertices $\{v_1, \ldots, v_n\}$, in which all edges are multiedges, we associate a set of nonzero vectors $\vec{V} = \{\vec{v_1}, \ldots, \vec{v_n}\}$ in $\mathbb{C}^n$ to the vertices of $G$ in the following manner: If vertices $v_i$ and $v_j$ are not joined then the corresponding vectors $\vec{v_i}$ and $\vec{v_j}$ are orthogonal. The rank of a vector representation $\vec{V}$ is the maximum number of linearly independent vectors in $\vec{V}$. The *minimum vector rank* of $G$, $mvr(G)$, is the minimum rank among all vector representations of $G$.

We present methods for determining $mvr(G)$ if $G$ is among certain classes of graphs, including perfect graphs and cycles. Further, we present upper and lower bounds on $mvr(G)$ for all multigraphs that contain only multiedges, and provide a conjecture on the exact value of $mvr(G)$ for such multigraphs.

In an appendix we also provide a conjecture about the minimum semidefinite rank of a graph $G$ in relation to the OS-number of $G$, as well as demonstrating a method that suggests the equality of these two values.

## 1 Introduction

A *multigraph* $G$ consists of a vertex set $V(G) = \{v_1, \ldots, v_n\}$ and an edge set $E(G)$ with multiple edges having the same pair of endpoints. Given a multigraph $G$, we associate a set of nonzero vectors $\{\vec{v_1}, \ldots, \vec{v_n}\}$ in $\mathbb{C}^n$ to its vertices such that if:

- $(v_i, v_j) \notin E(G)$ then $< \vec{v_i}, \vec{v_j} >= 0$

- $(v_i, v_j)$ is a single edge then $< \vec{v_i}, \vec{v_j} > \neq 0$
- $(v_i, v_j)$ is a multiple edge then $< \vec{v_i}, \vec{v_j} > \in \mathbb{C}$

where $< \cdot, \cdot >$ denotes the Euclidean inner product in $\mathbb{C}^n$. A set of vectors $\vec{V}$ satisfying the above conditions is said to be a *vector representation of* $G$ and the maximum number of linearly independent vectors in $\vec{V}$ is defined to be the *rank* of $\vec{V}$.

**Definition 1.1.** Given a multigraph $G$, the *minimum vector rank* of G, $\mathrm{mvr}(G)$, is defined as

$$\mathrm{mvr}(G) = \min\{\mathrm{rank}\,\vec{V} : \vec{V} \text{ is a vector representation of } G\}.$$

In this paper, we find $\mathrm{mvr}(G)$ for various classes of multigraphs $G$ where all edges are assumed to be multiple edges. Moreover, we present a conjecture for $\mathrm{mvr}(G)$ for all such multigraphs.

## 2 Results on $\mathrm{mvr}(G)$

Let $G$ be a multigraph with only multiple edges. Throughout we will use graph theory notation and definitions taken from [7].

**Definition 2.1.** A subset $V'$ of $V(G)$ is an *independent set* in $G$ if for all $u, v \in V', (u, v) \notin E(G)$.

**Definition 2.2.** The *independence number* $\alpha(G)$ is the size of a largest independent set.

**Lemma 2.3.** *Given a multigraph $G$, $\alpha(G) \leq \mathrm{mvr}(G)$.*

*Proof.* Let $V'$ be a largest independent set in $G$. Since any two vertices in $V'$ are not adjacent, the vectors corresponding to those vertices in any vector representation $\vec{V}$ of $G$ are orthogonal. Hence $\mathrm{rank}\,\vec{V} \geq \alpha(G)$. Thus, $\mathrm{mvr}(G) \geq \alpha(G)$. □

The inequality in the above lemma can be strict. For example for $C_5$, a cycle on five vertices, $\alpha(C_5) = 2$, but we will show in Proposition 2.16 that $\mathrm{mvr}(C_5) = 3$.

**Definition 2.4.** Suppose $\mathcal{C}$ is a collection of vertex disjoint cliqes of a multigraph $G$. We say that $\mathcal{C}$ is a *clique vertex cover* of $G$ if every vertex of $G$ belongs to exactly one clique in $\mathcal{C}$.

**Definition 2.5.** For a multigraph $G$ we define $\theta(G)$, the *clique vertex cover number* of $G$ as:

$$\theta(G) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is a clique vertex cover of } G\}.$$

**Lemma 2.6.** *For a multigraph $G$, $\mathrm{mvr}(G) \leq \theta(G)$.*

*Proof.* Let $\mathcal{C}$ be a clique vertex cover of $G$ such that $|\mathcal{C}| = \theta(G)$. For each clique $\mathcal{C}_i \in \mathcal{C}$ associate with it the standard basis vector $\vec{e_i}$. Consider a vector representation in which for each $v \in C_i$ in $\mathcal{C}$ take $\vec{v} = \vec{e_i}$. Since cliques in $\mathcal{C}$ are vertex disjoint, this assignment is well defined. Thus we get a vector representation $\vec{V}$ such that $\operatorname{rank} \vec{V} = \theta(G)$. Hence $\operatorname{mvr}(G) \leq \theta(G)$. □

At this time we do not have an example where the inequality in Lemma 2.6 is strict. In general, $\alpha(G) \leq \theta(G)$ and $C_5$ is an example where strict inequality occurs, since $\alpha(C_5) = 2 < 3 = \theta(C_5)$.

A *vector coloring* of a graph $G$ on $n$ vertices is an assignment of vectors in $\mathbb{C}^n$ to the vertices of $G$ such that vectors assigned to the endpoints of an edge are orthogonal. We denote by $\chi(G)$ the minimum number of vectors needed to vector color the graph $G$. Though this notion is the same as coloring vertices and chromatic number, we prefer the notion of vector coloring because of the following result.

**Lemma 2.7.** *Suppose $G$ is a graph and $G^c$ is a graph with only multiple edges, then* $\operatorname{mvr}(G^c) \leq \chi(G)$.

*Proof.* A vector coloring of $G$ produces a vector representation of $G^c$. Suppose $\vec{V}$ is a vector representation of $G^c$ corresponding to a vector coloring of $G$ with $\chi(G)$ vectors. Then $\operatorname{rank} \vec{V} \leq \chi(G)$. Hence $\operatorname{mvr}(G^c) \leq \chi(G)$. □

Notice that any (vector) coloring of $G$ partitions the vertices of $G^c$ into cliques. Thus $\theta(G^c) = \chi(G)$.

**Definition 2.8.** A subset of vertices $V'$ in a graph $G$ is said to be a *clique* in $G$ if for all $u, v \in V', u \neq v, (u, v) \in E(G)$. The *clique number* of $G$, $\omega(G)$, is defined as:

$$\omega(G) = \max\{|V'| : V' \subseteq V \text{ and } V' \text{ is a clique in } G\}.$$

*Observation* 2.9. In general $\omega(G) \leq \chi(G)$ and $\alpha(G^c) = \omega(G)$.

Since all the above notions relate only to subsets of vertices, they remain the same even when we consider graphs with only multiple edges. Therefore we will assume that all graphs and their complements have only single edges.

**Definition 2.10.** A graph $G$ is *perfect* if for all induced subgraphs $H$ of $G$, $\chi(H) = \omega(H)$.

The **Perfect Graph Theorem** states that "$G$ is perfect if and only if $G^c$ is perfect" [7]. Using this theorem and our earlier results we obtain the following:

**Theorem 2.11.** *If $G$ is a perfect graph then* $\operatorname{mvr}(G) = \chi(G^c) = \alpha(G)$.

*Proof.* Since $G$ is perfect, from the Perfect Graph Theorem $G^c$ is perfect. Hence $\chi(G^c) = \omega(G^c)$. From Lemma 2.7 we find that $\operatorname{mvr}(G) \leq \chi(G^c)$. From Lemma 2.3 we get $\alpha(G) \leq \operatorname{mvr}(G)$. Using Observation 2.9 we find that $\alpha(G) = \omega(G^c)$. Hence $\alpha(G) \leq \operatorname{mvr}(G) \leq \chi(G^c) = \omega(G^c) = \alpha(G)$, proving the theorem. □

The **Strong Perfect Graph Theorem** states that a graph $G$ is perfect if and only if both $G$ and $G^c$ have no induced subgraph that is an odd cycle of length at least five [2]. Using this theorem we find that many well-known classes of graphs are perfect. Hence we have the following list of corollaries.

**Corollary 2.12.** *If $G$ is a chordal graph with only multiple edges,* $\mathrm{mvr}(G) = \alpha(G)$. *In particular,* $\mathrm{mvr}(G) = 1$ *if and only if* $G = K_n$.

*Proof.* The first part follows from the fact that chordal graphs are perfect. Since complete graphs are chordal, this means $G = K_n \Rightarrow \mathrm{mvr}(G) = \alpha(G) = 1$. If $\mathrm{mvr}(G) = 1$, then there exists a vector representation containing only one linearly independent vector and multiples of it. Therefore all vertices in $G$ must be connected since all the vectors have a nonzero inner product. ☐

**Corollary 2.13.** *If $G$ is a bipartite graph with only multiple edges,* $\mathrm{mvr}(G) = \alpha(G)$. *Moreover,* $\mathrm{mvr}(K_{m,n}) = \max\{m, n\}$.

**Corollary 2.14.** *If $G$ is a path $P_n$ on $n$ vertices, then* $\mathrm{mvr}(P_n) = \lceil \frac{n}{2} \rceil$.

*Proof.* Alternating vertices on the path forms a maximal independent set of size $\lceil \frac{n}{2} \rceil$. ☐

**Corollary 2.15.** *If $C_n$ is an even cycle, then* $\mathrm{mvr}(C_n) = \frac{n}{2}$.

*Proof.* Observe that $C_n$ is bipartite and has two disjoint independent sets of size $\frac{n}{2}$. ☐

The following result shows that when $C_n$ is an odd cycle, which is an imperfect graph when $n \geq 5$, $\mathrm{mvr}(C_n)$ equals the chromatic number of its complement.

**Proposition 2.16.** *Let $C_n$ be an odd cycle with only multiple edges. Then* $\mathrm{mvr}(C_n) = \lceil \frac{n}{2} \rceil = \theta(C_n)$.

*Proof.* Let the number of vertices be $2k+1$. There is a minimal clique vertex cover of $C_n$ with k edges and one isolated vertex. Therefore assigning the standard basis vectors $\vec{e_1}, \ldots, \vec{e_k}$ in $\mathbb{C}^n$ to the $k$ edges and $\vec{e_{k+1}}$ to the isolated vertex we see that $\mathrm{mvr}(C_n) \leq k+1$.

Suppose $\mathrm{mvr}(C_n) = k$. Let $\{\vec{v_1}, \ldots, \vec{v_k}, \vec{v_{k+1}}, \ldots, \vec{v_{2k+1}}\}$ be a vector representation of rank $k$ with $\{\vec{v_1}, \ldots, \vec{v_k}\}$ linearly independent. If other vectors are linear combinations of $\vec{v_1}, \ldots, \vec{v_k}$, it follows that there is at least one of the $k+1$ cliques in the minimal clique vertex covering that has only linearly dependent vectors associated to its vertices. Since the vector associated to a vertex in this special clique has nonzero inner product with a basis vector in another clique, it creates a chord in $C_n$, which is not possible. Hence $\mathrm{mvr}(C_n) \geq k+1$. ☐

Of the 996 nonisomorphic connected graphs on seven or less vertices listed in [6], only 137 are not perfect. Among these it can be shown that $\mathrm{mvr}(G) = \chi(G^c)$ for certain classes of graphs, including for example odd cycles. Further, in all examples we have examined thus far, this equality holds. Other properties of these two quantities, as well as parallels between the construction of graph colorings and vector representations, suggest that it will hold for all graphs. So far, a proof of this fact has been elusive, leading to the following conjecture, which will guide future research efforts:

**Conjecture 2.17.** *If $G$ is a graph with only multiple edges then* $\text{mvr}(G) = \theta(G) = \chi(G^c)$.

# 3 Appendix: Is $\text{msr}(G) = \text{OS}(G)$?

**Definition 3.1** ([3]). Let $G$ be a simple graph and let $S = \{v_1, \ldots, v_m\}$ be an ordered set of vertices of $G$. Denote by $G_k$ the subgraph of $G$ induced by $\{v_1, \ldots, v_k\}$ for $k \leq m$. Let $H_k$ be the connected component of $G_k$ containing $v_k$. If for each $k$ there exists a vertex $w_k$ of $G$ such that $w_k \neq v_l, l \leq k, (w_k, v_k) \in E(G)$, and $(w_k, v_l) \notin E(G)$ for all $v_l$ in $H_k$ with $l \neq k$, we say $S$ is an *OS-vertex set* of $G$.

**Definition 3.2** ([3]). The maximum cardinality among all OS-vertex sets of $G$ is called the *OS-number* of $G$, written $\text{OS}(G)$.

Figure 1 shows a maximal OS-set for a graph, with $\{v_1, \ldots, v_m\}$ labelled and $\{w_1, \ldots, w_m\}$ labelled in parentheses.

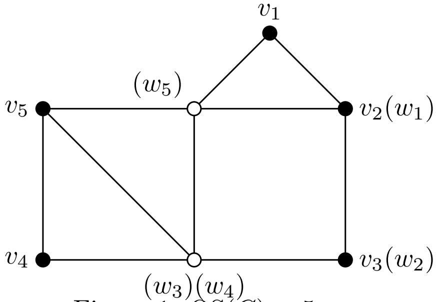

Figure 1: $\text{OS}(G) = 5$

Vertices $v_1, \ldots, v_n$ in a graph $G$ are called linearly independent if in any vector representation $\vec{V}$ of $G$, $\vec{v_1}, \ldots, \vec{v_n}$ are linearly independent vectors. The requirements for a set of vertices to be an OS-set ensure that the vertices of an OS-set are linearly independent, giving us the following theorem:

**Theorem 3.3** ([3]). *The minimum semidefinite rank of a graph $G$ is at least* $\text{OS}(G)$.

Whether or not $\text{OS}(G)$ is also an upper bound for $\text{msr}(G)$ remains an open question. It has been verified that $\text{OS}(G) = \text{msr}(G)$ for all graphs on seven or less vertices, and evidence suggests that this will be true for all graphs. For definitions and more information on the minimum semidefinite rank of a graph, see [1].

Consider a maximal OS-set $S = \{v_1, \ldots, v_m\}$ of a graph $G$ on $n$ vertices, and let $G_m$ be the subgraph of $G$ induced by the vertices in $S$. Label the vertices in $V(G) \backslash S$ as $v_{m+1}, \ldots, v_n$. Let $L(G_m)$ denote the Laplacian of $G_m$, and $I_m$ denote the $m \times m$ identity matrix. Now, define $A_{11} = L(G_m) + I_m$.

**Lemma 3.4.** $\operatorname{rank} A_{11} = \mathrm{OS}(G)$

*Proof.* It is known that the multiplicity of the zero eigenvalue of the Laplacian of a graph equals the number of connected components of the graph. Further, the nullity of the Laplacian equals the multiplicity of the zero eigenvalue, therefore $\operatorname{rank} L(G_m)$ equals $m$ minus the number of connected components in $G_m$.

Suppose $G_m$ is connected. Since $A_{11}$ is Hermitian and has positive leading principal minors, $A_{11}$ is positive definite [4]. Therefore $\operatorname{rank} A_{11} = m$

If $G_m$ is not connected, then $L(G_m)$ is a block matrix composed of disjoint blocks which are the Laplacians of the connected components of $G_m$. Figure 2 shows an example of this with two connected components $G_{m,1}$ and $G_{m,2}$. As per the previous argument, each of these block will have full rank. The total rank of $A_{11}$ is the sum of the ranks of the blocks, giving $\operatorname{rank} A_{11}$=m. □

$$L(G_m) = \left[\begin{array}{c|c} L(G_{m,1}) & 0 \\ \hline 0 & L(G_{m,2}) \end{array}\right]$$

Figure 2: Block form of $L(G_m)$

Place the matrix $A_{11}$ in the upper left of an $n \times n$ block matrix. Label the upper right block as $A_{12}$, the lower left as $A_{21}$, and the lower right as $A_{22}$ (See Figure 3). Call this block matrix $A$.

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{11}{}^*B^* \\ \hline BA_{11} & BA_{11}{}^*B^* \end{array}\right]$$

Figure 3: Construction of the matrix $A$

Call the rows of $A_{11}$ as $\vec{r_1}, \ldots, \vec{r_m}$. We can use linear combinations of $\vec{r_1}, \ldots, \vec{r_m}$ to construct the rows of $A_{21}$ . Say that the first row of $A_{21}$ is $c_{11}\vec{r_1} + \ldots + c_{1m}\vec{r_m}$, the second row of $A_{21}$ is $c_{21}\vec{r_1} + \ldots + c_{2m}\vec{r_m}$ and so on.

Then if we let $B$ be an $n-m\times m$ matrix $[b_{ij}]$ with $b_{ij}=c_{ij}$ we have that $A_{21} = BA_{11}$ . We want $A$ to be hermitian, so $A_{12} = A_{21}{}^* = A_{11}{}^*B^*$. Let $A_{22}$ equal $BA_{12} = BA_{11}{}^*B^*$.

**Lemma 3.5.** *$A$ is positive semidefinite.*

*Proof.* Since $A_{11}$ is positive semidefinite, we write $A_{11} = C^*C$ for some matrix $C$. Then we can see that:

$$\text{A}=\left[\begin{array}{c|c} C^*C & C^*CB^* \\ \hline BC^*C & BC^*CB^* \end{array}\right] = \left[\begin{array}{c} I_m \\ \hline B \end{array}\right] C^*C \left[\begin{array}{c|c} I_m & B^* \end{array}\right], \text{ so}$$

Letting $D = C\left[\begin{array}{c|c} I_m & B^* \end{array}\right]$ we see that $A = D^*D$. Hence $A$ is positive semidefinite.

Since $A$ can be decomposed into the form $A = D^*D$, $A$ is PSD. ☐

**Definition 3.6** ([5])**.** Given a block matrix $P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ the *Schur complement of $A$ in $P$* is denoted by $(P|A)$ and defined as:

$$(P|A) = D - CA^{-1}B.$$

*Remark* 3.7. It is known that $\operatorname{rank} P = \operatorname{rank} A + \operatorname{rank}(P|A)$.

**Lemma 3.8.** $\operatorname{rank} A = \mathrm{OS}(G)$.

*Proof.* $(A|A_{11}) = BA^*_{21} - BA_{11}A^{-1}_{11}A^*_{21} = BA^*_{11}B^* - BA_{11}A^{-1}_{11}A^*_{11}B^* = 0$. Then $\operatorname{rank} A = \operatorname{rank} A_{11} + \operatorname{rank}(A|A_{11}) = \operatorname{rank} A_{11}$. Since we know that $\operatorname{rank} A_{11} = \mathrm{OS}(G)$, we have that $\operatorname{rank} A = \mathrm{OS}(G)$. ☐

Thus any matrix $A$ constructed in this manner is positive semidefinite and has rank equal to $\mathrm{OS}(G)$. In order for $A$ to represent the graph $G$, for any $a_{ij}$ in $A_{12}, A_{21}$, or $A_{22}, a_{ij}$ must be zero if and only if vertices $v_i$ and $v_j$ are not joined in $G$. If there exists a matrix $B$ which will cause this condition to be satisfied in $A$, then we have a positive semidefinite matrix with rank equal to $\mathrm{OS}(G)$, so $\mathrm{msr}(G) = \mathrm{OS}(G)$.

In every example graph examined thus far, such a matrix $B$ does exist. Whether or not a suitable $B$ will always exist is an open question, as no proof has yet been found to show that it will, nor has a counterexample been discovered. This question is therefore of great interest for future study.

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# References

[1] Matthew Booth, Phil Hackney, Benjamin Harris, Charles R. Johnson, Margaret Lay, Lon H. Mitchell, Sivaram K. Narayan, Amanda Pascoe, and Brian D. Sutton, *The minimum semidefinite rank of a graph*, Manuscript.

[2] Reinhard Diestel, *Graph theory*, Springer, New York, 2006.

[3] Phil Hackney, Benjamin Harris, Margaret Lay, Lon H. Mitchell, Sivaram K. Narayan, and Amanda Pascoe, *Linearly independent vertices and minimum semidefinite rank*, Manuscript.

[4] Roger A. Horn and Charles R. Johnson, *Matrix analysis*, Cambridge University Press, New York, 1985.

[5] Viktor V. Prasolov, *Problems and theorems in linear algebra*, American Mathematical Society, Providence, RI, 1994.

[6] Ronald C. Read and Robin J. Wilson, *An atlas of graphs*, Oxford Science Publications, New York, 1998.

[7] Douglas B. West, *Introduction to graph theory*, Prentice Hall, Upper Saddle River, NJ, 2001.